Solutions to exercises in Section 2.9

Exercise 1 (Stieltjes transform and moments)**.** *Show that the Stieltjes transform $m_\mu(z)$ defined in Definition 3, of a probability measure μ with bounded support (and thus finite moments), is a moment generating function in the sense that, for all $z \in \mathbb{C}$ such that $|z| > \max\{|\inf(\operatorname{supp}(\mu))|, |\sup(\operatorname{supp}(\mu))|\}$,*

$$m_\mu(z) = -\frac{1}{z}\sum_{k=0}^{\infty} M_k z^{-k}$$

where $M_k = \int t^k \mu(dt)$.

From this formulation, propose a method to evaluate the successive moments of μ using m_μ.

Correction 1 (Stieltjes transform and moments)**.** *Note that*

$$\begin{aligned} m_\mu(z) &= \int \frac{1}{t-z}\mu(dt) = -\frac{1}{z}\int \frac{1}{1-t/z}\mu(dt) \\ &= -\frac{1}{z}\sum_{k=0}^{\infty}\int \frac{t^k}{z^k}\mu(dt) = -\frac{1}{z}\sum_{k=0}^{\infty} M_k z^{-k}. \end{aligned}$$

Thus, differentiating successively $\delta(z) \equiv -\frac{1}{z}m_\mu(z^{-1}) = \int \frac{1}{1-zt}\mu(dt)$, we find

$$\frac{d^\ell}{dz^\ell}\delta(z) = \sum_{k\geq \ell} \frac{k!}{(k-\ell)!} M_k z^{k-\ell}.$$

In particular, letting $z = 0$ in the expression above, we obtain

$$\frac{d^\ell}{dz^\ell}\delta(0) = \ell! \cdot M_\ell.$$

Exercise 2 (Non-immediate Stieltjes transforms)**.** *Let $\mathbf{X} \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $\mathbf{Q}(z) = (\mathbf{X} - z\mathbf{I}_n)^{-1}$ its resolvent. Show that, for any $\mathbf{u} \in \mathbb{R}^n$ of unit norm $\|\mathbf{u}\| = 1$ and any $\mathbf{A}$ nonnegative definite and such that $\operatorname{tr}\mathbf{A} = 1$, the quantities $\mathbf{u}^\mathsf{T}\mathbf{Q}(z)\mathbf{u}$ and $\operatorname{tr}\mathbf{A}\mathbf{Q}(z)$ are also Stieltjes transform of probability measures.*

What are these measures and what are their supports?

Correction 2 (Non-immediate Stieltjes transforms)**.** *Denote* $\mathbf{X} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\mathsf{T}$ *the spectral decomposition of* $\mathbf{X}$*, with* $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_n] \in \mathbb{R}^{n\times n}$ *and* $\mathbf{\Lambda} = \operatorname{diag}\{\lambda_i\}_{i=1}^n$*,* λ_i *the eigenvalues of* $\mathbf{X}$*. For the quantity* $\mathbf{u}^\mathsf{T}\mathbf{Q}(z)\mathbf{u}$*, it suffices to observe that, for* $\mathbf{w} = \mathbf{V}^\mathsf{T}\mathbf{u}$*,*

$$\mathbf{u}^\mathsf{T}\mathbf{Q}(z)\mathbf{u} = \mathbf{u}^\mathsf{T}\mathbf{V}\operatorname{diag}\left\{\frac{1}{\lambda_i - z}\right\}_{i=1}^n \mathbf{V}^\mathsf{T}\mathbf{u} = \sum_{i=1}^n \frac{[\mathbf{w}]_i^2}{\lambda_i - z} = \int \frac{1}{t-z}\mu(dt),$$

where we define $\mu = \sum_{i=1}^n [\mathbf{w}]_i^2 \delta_{\lambda_i}$*. In particular,*

$$\int \mu(dt) = \sum_{i=1}^n [\mathbf{w}]_i^2 = \mathbf{w}^\mathsf{T}\mathbf{w} = \mathbf{u}^\mathsf{T}\mathbf{V}\mathbf{V}^\mathsf{T}\mathbf{u} = 1,$$

so that, since $[\mathbf{w}]_i^2\delta_{\lambda_i}(t) \geq 0$*,* μ *is a probability measure.*

Letting similarly $\mathbf{B} = \mathbf{V}^\mathsf{T}\mathbf{A}\mathbf{V}$*,*

$$\operatorname{tr}\mathbf{A}\mathbf{Q} = \operatorname{tr}\mathbf{B}\operatorname{diag}\left\{\frac{1}{\lambda_i - z}\right\}_{i=1}^n = \sum_{i=1}^n \frac{[\mathbf{B}]_{ii}}{\lambda_i - z} = \int \frac{1}{t-z}\nu(dt),$$

where we define $\nu = \sum_{i=1}^n [\mathbf{B}]_{ii}\delta_{\lambda_i}$*, with* $[\mathbf{B}]_{ii} \geq 0$ *(to see this, write* $\mathbf{A} = \mathbf{V}'\mathbf{\Lambda}'(\mathbf{V}')^\mathsf{T}$*, so that* $[\mathbf{B}]_{ii} = [\mathbf{V}^\mathsf{T}\mathbf{V}']_{i\cdot}\mathbf{\Lambda}'[(\mathbf{V}')^\mathsf{T}\mathbf{V}]_{\cdot i} \geq 0$ *since* $\mathbf{\Lambda}'$ *is a diagonal of non-negative entries) and* $\int \nu(dt) = \operatorname{tr}\mathbf{B} = \operatorname{tr}\mathbf{A} = 1$*.*

Note that the measures μ *and* ν *can be viewed as weighted versions of the empirical spectral measure of* $\mathbf{X}$*: for instance, taking* $\mathbf{w} = \frac{1}{\sqrt{n}}\mathbf{1}_n$ *and* $\mathbf{B} = \frac{1}{n}\mathbf{I}_n$ *one obtains the standard empirical spectral measure of* $\mathbf{X}$ *as defined in Definition 2. The support of both measures* μ *and* ν *is thus included in the support of the original empirical spectral measure* $\sum_{i=1}^n \delta_{\lambda_i}$ *(i.e.,* $\{\lambda_1, \ldots, \lambda_n\}$*). They however only coincide with the support if* $[\mathbf{w}]_i^2 \neq 0$ *for all* i*'s associated to a common* λ_j *(respectively,* $[\mathbf{B}]_{ii} \neq 0$*).*

Exercise 3 (Stieltjes transform and singular values)**.** *Let* μ *be a probability measure on* $\mathbb{R}^+$ *and* ν, ν' *be the measures defined by*

$$\int f(t)\nu(dt) = \int f(\sqrt{t})\mu(dt)$$
$$\int f(t)\nu'(dt) = \frac{1}{2}\left(\int f(t)\nu(dt) + \int f(-t)\nu(dt)\right)$$

for all bounded continuous f*.*

What are ν *and* ν' *when* $\mu = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i}$ *for some* $\lambda_1, \ldots, \lambda_n \geq 0$*?*

Show that the Stieltjes transform $m_{\nu'}$ *of* ν' *satisfies*

$$m_{\nu'}(z) = z m_\mu(z^2). \tag{8.2}$$

Letting $\mathbf{X} \in \mathbb{R}^{n\times p}$ *and* μ *be the empirical spectral measure of* $\mathbf{X}\mathbf{X}^\mathsf{T}$ *as in Definition 2, relate the Stieltjes transform of the empirical spectral measure of the matrix*

$$\mathbf{\Gamma} = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{X} \\ \mathbf{X}^\mathsf{T} & \mathbf{0}_{p\times p} \end{bmatrix} \in \mathbb{R}^{(n+p)\times(n+p)} \tag{8.3}$$

to that of the measure μ, and conclude on the nature of this Stieltjes transform for $n = p$.

Correction 3 (Stieltjes transform and singular values)**.** *For $\mu = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i}$, we have $\nu = \frac{1}{n}\sum_{i=1}^n \delta_{\sqrt{\lambda_i}}$ and $\nu' = \frac{1}{2n}\sum_{i=1}^n (\delta_{\sqrt{\lambda_i}} + \delta_{-\sqrt{\lambda_i}})$.*

We thus have

$$\begin{aligned} m_{\nu'}(z) &= \frac{1}{2}\int \frac{1}{t-z}\nu(dt) + \frac{1}{2}\int \frac{1}{-t-z}\nu(dt) \\ &= \frac{1}{2}\int \frac{1}{\sqrt{t}-z}\mu(dt) + \frac{1}{2}\int \frac{1}{-\sqrt{t}-z}\mu(dt) \\ &= z\int \frac{1}{t-z^2}\mu(dt) = z m_\mu(z^2). \end{aligned}$$

The Stieltjes transform of the empirical spectral measure of $\mathbf{\Gamma}$ is given by

$$\begin{aligned} \frac{1}{n+p}\operatorname{tr}\begin{pmatrix} -z\mathbf{I}_n & \mathbf{X} \\ \mathbf{X}^\mathsf{T} & -z\mathbf{I}_p \end{pmatrix}^{-1} &= \frac{z}{n+p}\operatorname{tr}(\mathbf{X}\mathbf{X}^\mathsf{T} - z^2\mathbf{I}_n)^{-1} + \frac{z}{n+p}\operatorname{tr}(\mathbf{X}^\mathsf{T}\mathbf{X} - z^2\mathbf{I}_p)^{-1} \\ &= \frac{2z}{n+p}\operatorname{tr}(\mathbf{X}\mathbf{X}^\mathsf{T} - z^2\mathbf{I}_n)^{-1} + \frac{n-p}{n+p}\frac{1}{z}, \end{aligned}$$

where we used the block-matrix inverse lemma, Lemma 2.5, and then Lemma 2.4. This Stieltjes transform has its singularities for z the singular values of $\mathbf{X}$ (i.e., the square-root of the eigenvalues of $\mathbf{X}\mathbf{X}^\mathsf{T}$) and thus relates to the singular spectrum of $\mathbf{X}$, as well as for $z = 0$ when $n \neq p$. In particular, taking $n = p$ (so that $\mathbf{X}$ is a square matrix), we recover the relation in (8.2).

Exercise 4 (Proof of Lemma 2.9: a special case)**.** *For $\mathbf{A}, \mathbf{M} \in \mathbb{R}^{p\times p}$ symmetric nonnegative definite matrices, $\mathbf{u} \in \mathbb{R}^p$, $\tau > 0$ and $z < 0$, show that*

$$\left| \operatorname{tr}\mathbf{A}\left(\mathbf{M} + \tau\mathbf{u}\mathbf{u}^\mathsf{T} - z\mathbf{I}_p\right)^{-1} - \operatorname{tr}\mathbf{A}\left(\mathbf{M} - z\mathbf{I}_p\right)^{-1} \right| \leq \frac{\|\mathbf{A}\|}{|z|}.$$

Correction 4 (Proof of Lemma 2.9: a special case)**.** *Using the resolvent identity, Lemma 2.1, and Sherman–Morrison formula, Lemma 2.8, we write*

$$\begin{aligned} &\left| \operatorname{tr}\mathbf{A}\left(\mathbf{M} + \tau\mathbf{u}\mathbf{u}^\mathsf{T} - z\mathbf{I}_p\right)^{-1} - \operatorname{tr}\mathbf{A}\left(\mathbf{M} - z\mathbf{I}_p\right)^{-1} \right| \\ &= \left| \operatorname{tr}\mathbf{A}\left(\mathbf{M} + \tau\mathbf{u}\mathbf{u}^\mathsf{T} - z\mathbf{I}_p\right)^{-1} \tau\mathbf{u}\mathbf{u}^\mathsf{T}\left(\mathbf{M} - z\mathbf{I}_p\right)^{-1} \right| \\ &= \frac{\left| \operatorname{tr}\mathbf{A}\left(\mathbf{M} - z\mathbf{I}_p\right)^{-1} \tau\mathbf{u}\mathbf{u}^\mathsf{T}\left(\mathbf{M} - z\mathbf{I}_p\right)^{-1} \right|}{1 + \tau\mathbf{u}^\mathsf{T}\left(\mathbf{M} - z\mathbf{I}_p\right)^{-1}\mathbf{u}} \\ &= \frac{\left| \tau\mathbf{u}^\mathsf{T}\left(\mathbf{M} - z\mathbf{I}_p\right)^{-1}\mathbf{A}\left(\mathbf{M} - z\mathbf{I}_p\right)^{-1}\mathbf{u} \right|}{1 + \tau\mathbf{u}^\mathsf{T}\left(\mathbf{M} - z\mathbf{I}_p\right)^{-1}\mathbf{u}}. \end{aligned}$$

At this point, we use the fact that $|\mathbf{x}^\mathsf{T}\mathbf{B}\mathbf{x}| \leq \|\mathbf{B}\| \cdot \|\mathbf{x}\|^2$ with $\mathbf{B} = (\mathbf{M} - z\mathbf{I}_p)^{-\frac{1}{2}}\mathbf{A}(\mathbf{M} - z\mathbf{I}_p)^{-\frac{1}{2}}$ (note the important for z to be real negative to ensure *that* $\mathbf{B}$ *is symmetric and nonnegative definite) and* $\mathbf{x} = (\mathbf{M} - z\mathbf{I}_p)^{-\frac{1}{2}}\mathbf{u}$*, so to obtain*

$$\begin{aligned}&\left|\operatorname{tr}\mathbf{A}\left(\mathbf{M} + \tau\mathbf{u}\mathbf{u}^\mathsf{T} - z\mathbf{I}_p\right)^{-1} - \operatorname{tr}\mathbf{A}\left(\mathbf{M} - z\mathbf{I}_p\right)^{-1}\right| \\ &\leq \|(\mathbf{M} - z\mathbf{I}_p)^{-\frac{1}{2}}\mathbf{A}(\mathbf{M} - z\mathbf{I}_p)^{-\frac{1}{2}}\| \cdot \frac{\tau\mathbf{u}^\mathsf{T}(\mathbf{M} - z\mathbf{I}_n)^{-1}\mathbf{u}}{1 + \tau\mathbf{u}^\mathsf{T}(\mathbf{M} - z\mathbf{I}_n)^{-1}\mathbf{u}}.\end{aligned}$$

Using finally $\|(\mathbf{M} - z\mathbf{I}_p)^{-\frac{1}{2}}\mathbf{A}(\mathbf{M} - z\mathbf{I}_p)^{-\frac{1}{2}}\| \leq \|\mathbf{A}\| \cdot \|(\mathbf{M} - z\mathbf{I}_n)^{-1}\|$, $\|(\mathbf{M} - z\mathbf{I}_n)^{-1}\| \leq |z|^{-1}$ *and* $\frac{x}{1+x} \leq 1$ *for* $x > 0$*, we conclude the proof.*[4]

Exercise 5 (Proof of Nash–Poincaré inequality, Lemma 2.14)**.** *The objective of the exercise is to show that, for* $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$ *with* $\mathbf{C} \in \mathbb{R}^{p\times p}$ *and* $f : \mathbb{R}^p \to \mathbb{R}$ *of bounded first- and second-order derivatives,*

$$\operatorname{Var}[f(\mathbf{x})] \leq \mathbb{E}\left[(\nabla f(\mathbf{x}))^\mathsf{T}\mathbf{C}\nabla f(\mathbf{x})\right].$$

To this end, it is convenient to first define an "interpolating" Gaussian vector $\mathbf{x}(t) = \sqrt{t}\mathbf{x}_1 + \sqrt{1-t}\mathbf{x}_2$ *for* $t \in [0, 1]$ *with* $\mathbf{x}_1 \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_1)$*,* $\mathbf{x}_2 \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_2)$ *independent, and show, by applying successively the chain rule and Stein's lemma, Lemma 2.13, that for twice differentiable* g*,*

$$\begin{aligned}\mathbb{E}[g(\mathbf{x}_1)] - \mathbb{E}[g(\mathbf{x}_2)] &= \int_0^1 \frac{d}{dt}\mathbb{E}[g(\mathbf{x}(t))]\,dt \\ &= \frac{1}{2}\int_0^1 \mathbb{E}\left[\nabla g(\mathbf{x}(t))^\mathsf{T}\mathbf{C}_1\nabla g(\mathbf{x}(t)) - \nabla g(\mathbf{x}(t))^\mathsf{T}\mathbf{C}_2\nabla g(\mathbf{x}(t))\right]dt.\end{aligned}$$

From there, apply the result to the vectors $\mathbf{x}_1 = [\mathbf{y}^\mathsf{T}, \mathbf{y}^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{2p}$ *and* $\mathbf{x}_2 = [\mathbf{y}_1^\mathsf{T}, \mathbf{y}_2^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{2p}$ *for* $\mathbf{y}, \mathbf{y}_1, \mathbf{y}_2 \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$ *independent, and* $g([\mathbf{a}^\mathsf{T}, \mathbf{b}^\mathsf{T}]^\mathsf{T}) = f(\mathbf{a})f(\mathbf{b})$*. Conclude by an application of Cauchy-Schwarz inequality on the expectation under the resulting integrand and the observation that the bound on the integrand is constant with respect to* t*.*

Correction 5 (Proof of Nash–Poincaré inequality, Lemma 2.14)**.** *Here we follow the proof approach in [Pastur, 2005] and show the following slightly more general result: for* $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$ *with* $\mathbf{C} \in \mathbb{R}^{p\times p}$ *and* $f_1, f_2 : \mathbb{R}^p \to \mathbb{R}$ *both of bounded first- and second-order derivatives,*

$$\operatorname{Cov}[f_1(\mathbf{x}), f_2(\mathbf{x})] \leq \sqrt{\mathbb{E}[(\nabla f_1(\mathbf{x}))^\mathsf{T}\mathbf{C}\nabla f_1(\mathbf{x})]} \cdot \sqrt{\mathbb{E}[(\nabla f_2(\mathbf{x}))^\mathsf{T}\mathbf{C}\nabla f_2(\mathbf{x})]}. \quad (8.4)$$

Taking $f_1 = f_2 = f$ *falls back on the result of Lemma 2.14.*

For independent Gaussian random vectors $\mathbf{x}_1 \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_1)$ *and* $\mathbf{x}_2 \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_2)$*,* $\mathbf{C}_1, \mathbf{C}_2 \in \mathbb{R}^{N\times N}$*, define an "interpolating" Gaussian vector* $\mathbf{x}(t)$ *as*

$$\mathbf{x}(t) = \sqrt{t}\mathbf{x}_1 + \sqrt{1-t}\mathbf{x}_2, \quad t \in [0, 1].$$

[4] The proof of the result for z complex with positive imaginary part is available in the original article of [Silverstein and Bai, 1995].

Then, for $g : \mathbb{R}^N \to \mathbb{R}$ with bounded first and second-order derivatives, by the chain rule of derivation,

$$\begin{aligned}
\mathbb{E}[g(\mathbf{x}_1)] - \mathbb{E}[g(\mathbf{x}_2)] &= \int_0^1 \frac{d}{dt}\mathbb{E}[g(\mathbf{x}(t))]\,dt \\
&= \int_0^1 \sum_{i=1}^N \mathbb{E}\frac{\partial g(\mathbf{x}(t))}{\partial[\mathbf{x}(t)]_i}\left[\frac{\mathbf{x}_1}{2\sqrt{t}} - \frac{\mathbf{x}_2}{2\sqrt{1-t}}\right]_i dt \\
&= \frac{1}{2}\int_0^1 \mathbb{E}\left[\nabla g(\mathbf{x}(t))^\mathsf{T}\mathbf{C}_1\nabla g(\mathbf{x}(t)) - \nabla g(\mathbf{x}(t))^\mathsf{T}\mathbf{C}_2\nabla g(\mathbf{x}(t))\right] dt \qquad (8.5)
\end{aligned}$$

where in the last line, we crucially used Stein's lemma, Lemma 2.13.[5] *In detail, when applying Stein's lemma, the partial derivative with respect to $\mathbf{x}(t)$ must be expanded (as per the chain rule of derivation) as a sum of partial derivatives with respect to $\mathbf{x}_1$ and $\mathbf{x}_2$: it appears (thanks to the prefactors $\sqrt{t}$ and $\sqrt{1-t}$) that the second-order derivatives vanish in the expansion and only the cross derivatives remain; and hence the final quadratic form in the gradients. To exploit this result (which already provides the desired quadratic forms on the gradients), we choose $N = 2p$, $g([\mathbf{x}^\mathsf{T}, \mathbf{y}^\mathsf{T}]^\mathsf{T}) = f_1(\mathbf{x})f_2(\mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$, and $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^{2p}$ in such as way that*

$$\mathbf{x}_1 = \begin{bmatrix}\mathbf{y}\\ \mathbf{y}\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix} \qquad (8.6)$$

for some Gaussian random vector $\mathbf{y} \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$ and $\mathbf{y}_1, \mathbf{y}_2$ independent *copies of $\mathbf{y}$. In particular, for these vectors,*

$$\mathbb{E}[\mathbf{x}_1\mathbf{x}_1^\mathsf{T}] \equiv \mathbf{C}_1 = \begin{bmatrix}\mathbf{C} & \mathbf{C}\\ \mathbf{C} & \mathbf{C}\end{bmatrix}, \quad \mathbb{E}[\mathbf{x}_2\mathbf{x}_2^\mathsf{T}] \equiv \mathbf{C}_2 = \begin{bmatrix}\mathbf{C} & \mathbf{0}_p\\ \mathbf{0}_p & \mathbf{C}\end{bmatrix}$$

and we have the difference of expectations

$$\mathbb{E}[g(\mathbf{x}_1)] - \mathbb{E}[g(\mathbf{x}_2)] = \mathbb{E}[f_1(\mathbf{y})f_2(\mathbf{y})] - \mathbb{E}[f_1(\mathbf{y}_1)]\mathbb{E}[f_2(\mathbf{y}_2)] \equiv \mathrm{Cov}[f_1(\mathbf{x}), f_2(\mathbf{x})]$$

for $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$. As a consequence, from (8.5), we find that

$$\mathrm{Cov}[f_1(\mathbf{x}), f_2(\mathbf{x})] = \int_0^1 \mathbb{E}\left[\nabla f_1(\hat{\mathbf{x}}(t))^\mathsf{T}\mathbf{C}\nabla f_2(\hat{\mathbf{y}}(t))\right] dt$$

for $\hat{\mathbf{x}}(t) \equiv \sqrt{t}\mathbf{y} + \sqrt{1-t}\mathbf{y}_1$ and $\hat{\mathbf{y}}(t) \equiv \sqrt{t}\mathbf{y} + \sqrt{1-t}\mathbf{y}_2$, $t \in [0, 1]$. The proof then concludes by noting that $\hat{\mathbf{x}}(t)$ and $\hat{\mathbf{y}}(t)$ are (zero-mean) identically distributed Gaussian random vectors and that their distribution only *depends on $\mathbf{C}$ and* not *on t: the result follows from applying Cauchy-Schwarz inequality to the integrand and realizing that the integral can be freely removed.*

[5] This in passing shows that, in essence, Stein's lemma alone can be used to derive random matrix asymptotics without explicitly resorting to the Nash-Poincaré inequality.

Exercise 6 (The $\sqrt{|x-E|}$ behavior of the edges)**.** *Show that both the semicircle law (Theorem 2.5) and the Marčenko-Pastur law (Theorem 2.4, for $c \neq 1$) have a local $\sqrt{|x-E|}$ behavior at each of the edges E of their support.*

Conclude on the typical number of eigenvalues of the Wishart matrix $\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T} \in \mathbb{R}^{p\times p}$ (with $\mathbf{X}_{ij} \sim \mathcal{N}(0,1)$ independent) and the Wigner $\frac{1}{\sqrt{n}}\mathbf{X} \in \mathbb{R}^{n\times n}$ (with say $\mathbf{X}_{ij} = \mathbf{X}_{ji} \sim \mathcal{N}(0,1)$ independent up to symmetry) found near the edges of their respective supports.

Relate this finding to the Tracy-Widom distribution of the fluctuations of the largest and smallest eigenvalues in Theorem 2.15.

What happens for the left-edge of the support of the Marčenko-Pastur law and to the associated smallest eigenvalues of Wishart matrices when $\lim p/n = c = 1$? How many eigenvalues are then found close to the left edge in this so-called "hard-edge" setting? Conclude on the typical fluctuations of these eigenvalues and confirm numerically.

Correction 6 (The $\sqrt{|x-E|}$ behavior of the edges)**.** *The Marčenko-Pastur density can be approximated around, say, the right-edge $E_+ = (1+\sqrt{c})^2$ (i.e., $x = (1+\sqrt{c})^2 - \epsilon$, $\epsilon > 0$ small) of its support as*

$$\mu(dx) \simeq_{x\uparrow E_+} \frac{\sqrt{(1+\sqrt{c})^2 - (1-\sqrt{c})^2}}{2\pi c(1+\sqrt{c})^2} \sqrt{|x-(1+\sqrt{c})^2|}\, dx.$$

Similarly, around $E_+ = 2$, the semicircle distribution has approximate density

$$\mu(dx) = \frac{1}{2\pi}\sqrt{(x^2-4)^+}dx \simeq_{x\uparrow E_+} \frac{1}{\pi}\sqrt{|x-2|}\, dx.$$

As a consequence, for both models, within the interval $[E_+ - \epsilon, E_+]$ of length ϵ small, the typical integral of the density is of order

$$\int_{E_+-\epsilon}^{E_+} \mu(dx) \simeq C \int_{E_+-\epsilon}^{E_+} \sqrt{E_+ - x}\, dx = C \int_0^\epsilon \sqrt{y}\, dy = \frac{2C}{3}\epsilon^{\frac{3}{2}},$$

with $y = E_+ - x$ and $C > 0$ containing the constants provided above.

As a consequence, near the edge E_+, to have an $O(1)$ number of eigenvalues lying within the interval $[E_+ - \epsilon, E_+]$, one needs to scale ϵ as $\epsilon = O(n^{-2/3})$ (so that $\int_{E_+-\epsilon}^{E_+} \mu(dx) = O(n^{-1})$). This is particularly interesting as, elsewhere in the support, the typical "spacing" between adjacent eigenvalues is of order $O(n^{-1})$ (in total n eigenvalues in an interval of size $O(1)$). This precisely means that the eigenvalues are "more spread out" to each other near the edges *with a spacing of order $O(n^{-2/3})$, which is (not surprisingly) also the typical fluctuation of the extreme eigenvalues according to the Tracy-Widom theorem, Theorem 2.15.*

When $c = 1$ in the Marčenko-Pastur case though, due to the presence of the term x in the denominator of the density $\mu(dx)$, we obtain instead that, for x near the left edge $E_- = 0$,

$$\mu(dx) \simeq_{x\downarrow 0} \frac{1}{\pi\sqrt{x}}\, dx,$$

so that the density diverges at zero and the integral of the density on $[0, \epsilon]$ is instead of order $O(\sqrt{\epsilon})$: consequently, the typical number of eigenvalues is here extremely large, of order n^2 (extremely locally of course).

Exercise 7 (The $\sqrt{|x-E|}$ behavior in elaborate models)**.** *We here seek to extend the results in Exercise 6 to the sample covariance matrix model $\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T}$ where $\mathbf{X} = \mathbf{C}^{\frac{1}{2}}\mathbf{Z}$ with $\mathbf{Z}$ having independent entries of zero mean, unit variance and $\mathbf{C}$ having a bounded limiting spectral measure ν with fast decaying tails. We denote $\tilde{m}(z)$ the Stieltjes transform of the limiting spectral measure $\tilde{\mu}$ of $\frac{1}{n}\mathbf{X}^\mathsf{T}\mathbf{X}$.*

Using Figure 2.5 as a reference and recalling the formulation for functional inverse

$$x(\tilde{m}) = -\frac{1}{\tilde{m}} + c\int \frac{t\nu(dt)}{1+t\tilde{m}} \tag{8.7}$$

extensively discussed in Section 2.3.1, visually justify that $x''(\tilde{m})$ can be (complex) analytically extended in the neighborhood of the local extrema of $x(\tilde{m})$ (that is, each point $\tilde{m}$ where $x'(\tilde{m}) = 0$) into a function $z(\tilde{m})$ which must locally coincide with the inverse Stieltjes transform of $\tilde{m}(z)$.

Deduce that $\tilde{m}(z)$ must be of the form $\sqrt{z-E}$ near an edge E and conclude.

Correction 7 (The $\sqrt{|x-E|}$ behavior in elaborate models)**.** *We observe visually from Figure 2.5 that $x''(\tilde{m}) \neq 0$ near the edges of the support (so that $x(\tilde{m})$ has a quadratic local behavior per Taylor expansion). As such, the function $z(\tilde{m})$, defined similar to $x(\tilde{m})$ but extended for $\tilde{m} \in \mathbb{C} \setminus \{-1/\operatorname{supp}(\nu)\}$, is analytic and locally invertible into a function $\tilde{M}(z)$. This function $\tilde{M}(z)$ coincides with the Stieltjes transform $\tilde{m}(z)$ of $\tilde{\mu}$ for all z real outside the support. By analyticity (in particular of all its derivatives), for z near the edges of the support (say for $z = x + \imath\varepsilon$ with x inside the support and $\varepsilon \ll 1$), $\tilde{m}(z)$ and its inverse $z(\tilde{m})$ must enjoy the same functional behavior: the latter is in particular locally quadratic, and thus, as its inverse, the former has a local square-root behavior. Since the density equals the imaginary part of $\tilde{m}(z)$ for z near the real axis, this square-root behavior propagates to the density which must then be of the form $\sqrt{z-E}$.*

In passing, note that we implicitly assumed the existence of asymptotes for $-1/\tilde{m} \in \operatorname{supp}(\nu)$ and the local extrema $\tilde{m}_0$ such that $x'(\tilde{m}_0) = 0$ (as in Figure 2.5), which, depending on the limiting spectral measure ν of $\mathbf{C}$, may not always be the case. In fact, unlike in Figure 2.5, $x(\tilde{m})$ may not *have asymptotes on the edges of its domain of definition (i.e., the set of $\tilde{m}$ such that $-1/\tilde{m} \notin \operatorname{supp}(\nu)$) when ν has* slow *decaying tails near an edge. This is what occurs in Figure 8.7 where $\mathbf{C}$ is of the form $\mathbf{C} = \frac{1}{n}\mathbf{D}^{\frac{1}{2}}\mathbf{Z}\mathbf{Z}^\mathsf{T}\mathbf{D}^{\frac{1}{2}}$ for $\mathbf{D}$ having a discrete limiting spectral measure and $\mathbf{Z}$ having i.i.d. $\mathcal{N}(0,1)$ entries; this example will be discussed in more detail in Exercise 8.*

Exercise 8 (Further results on $x(\tilde{m})$)**.** *We aim in this exercise to justify some of the visual observations in Figure 2.5 with the help of*

$$x(\tilde{m}) = -\frac{1}{\tilde{m}} + c\int \frac{t\nu(dt)}{1+t\tilde{m}}. \tag{8.8}$$

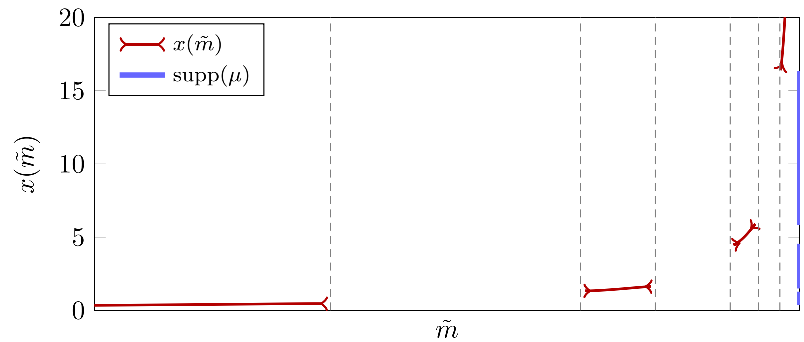

Figure 8.7: Functional inverse of $\tilde{m}(z)$, the Stieltjes transform of the limiting spectral measure of $\frac{1}{n}\mathbf{C}^{\frac{1}{2}}\mathbf{X}\mathbf{X}^{\mathsf{T}}\mathbf{C}^{\frac{1}{2}}$ with $\mathbf{C} = \frac{1}{n}\mathbf{D}^{\frac{1}{2}}\mathbf{Z}\mathbf{Z}^{\mathsf{T}}\mathbf{D}^{\frac{1}{2}}$ for $\mathbf{D} = \text{diag}[\mathbf{1}_{p/3},\ 3\cdot \mathbf{1}_{p/3},\ 10\cdot\mathbf{1}_{p/3}]$, independent Gaussian $\mathbf{X}, \mathbf{Z} \in \mathbb{R}^{p\times n}$ and $p/n = 1/10$. Link to code: Matlab and Python.

Show that, for $\tilde{m}_1 \neq \tilde{m}_2$ such that $x'(\tilde{m}_1), x'(\tilde{m}_2) > 0$, we cannot have $x(\tilde{m}_1) = x(\tilde{m}_2)$: that is, the increasing segments of $x(\tilde{m})$ never "overlap".

Besides, show that, if $\tilde{m}_1 < \tilde{m}_2$ are both of the same sign, and $x'(\tilde{m}_1), x'(\tilde{m}_2) > 0$, then $x(\tilde{m}_1) < x(\tilde{m}_2)$: that is, the increasing segments of $x(\tilde{m})$ never "swap". To this end, we may prove the intermediary result

$$(\tilde{m}_1 - \tilde{m}_2)\left(1 - \int \frac{c\tilde{m}_1\tilde{m}_2 t^2 \nu(dt)}{(1+t\tilde{m}_1)(1+t\tilde{m}_2)}\right) = \tilde{m}_1\tilde{m}_2(x(\tilde{m}_1) - x(\tilde{m}_2)) \tag{8.9}$$

and use Cauchy-Schwarz inequality to control the left-hand side term.

Finally show that, if ν has bounded support, then $x(\tilde{m}) \to 0$ as $\tilde{m} \to \pm\infty$.

As a final remark, note that the only important observation about Figure 2.5 which we have not shown here is the fact that the points $\tilde{m}$ where $x'(\tilde{m}) = 0$ must exist. In fact, this is not always the case and heavily depends on the nature of the tails of the measure ν. Justify in particular that, for some ν, there may be no asymptote on the edges of the domain of definition of $x(\cdot)$ (as opposed to what is seen in Figure 2.5).

Correction 8 (Further results on $x(\tilde{m})$)**.** *If $x(\tilde{m}_1) = x(\tilde{m}_2)$ for $\tilde{m}_1, \tilde{m}_2$ such that $x'(\tilde{m}_1), x'(\tilde{m}_2) > 0$, then $x(\tilde{m})$ is locally invertible around both $\tilde{m}_1$ and $\tilde{m}_2$ with inverse the Stieltjes transform $\tilde{m}(\cdot)$ of $\tilde{\mu}$. In particular, $\tilde{m}(x(\tilde{m}_1))$ and $\tilde{m}(x(\tilde{m}_2))$ are both uniquely defined (since the Stieltjes transform of a valid z outside the support is unique) and equal to $\tilde{m}_1 \neq \tilde{m}_2$, respectively. It is thus impossible that $x(\tilde{m}_1) = x(\tilde{m}_2)$ if $\tilde{m}_1 \neq \tilde{m}_2$.*

If $\tilde{m}_1 < \tilde{m}_2$ and $x'(\tilde{m}_1), x'(\tilde{m}_2) > 0$, then

$$x(\tilde{m}_1) - x(\tilde{m}_2) = (\tilde{m}_1 - \tilde{m}_2)\left[\frac{1}{\tilde{m}_1\tilde{m}_2} - c\int \frac{t^2\nu(dt)}{(1+t\tilde{m}_1)(1+t\tilde{m}_2)}\right],$$

or equivalently

$$(x(\tilde{m}_1) - x(\tilde{m}_2))\tilde{m}_1\tilde{m}_2 = (\tilde{m}_1 - \tilde{m}_2)\left[1 - c\int \frac{t^2\tilde{m}_1\tilde{m}_2\nu(dt)}{(1+t\tilde{m}_1)(1+t\tilde{m}_2)}\right].$$

In addition, observe that, taking the limit where $\tilde{m}_1 - \tilde{m}_2 \to 0$*, we have the derivative formulation*

$$x'(\tilde{m})\tilde{m}^2 = 1 - c\int \frac{t^2\tilde{m}^2\nu(dt)}{(1+t\tilde{m})^2},$$

which is positive for both $\tilde{m} = \tilde{m}_1$ *and* $\tilde{m} = \tilde{m}_2$*. Using Cauchy-Schwarz inequality, we thus find that*

$$\left|c\int \frac{t^2\tilde{m}_1\tilde{m}_2\nu(dt)}{(1+t\tilde{m}_1)(1+t\tilde{m}_2)}\right| \leq \sqrt{c\int \frac{t^2\tilde{m}_1^2\nu(dt)}{(1+t\tilde{m}_1)^2} \cdot c\int \frac{t^2\tilde{m}_2^2\nu(dt)}{(1+t\tilde{m}_2)^2}} < 1.$$

As a consequence,

$$(x(\tilde{m}_1) - x(\tilde{m}_2))\frac{\tilde{m}_1\tilde{m}_2}{\tilde{m}_1 - \tilde{m}_2} > 0.$$

Since $\tilde{m}_1\tilde{m}_2 > 0$*, we conclude that* $x(\tilde{m}_1) - x(\tilde{m}_2)$ *is of the same sign as* $\tilde{m}_1 - \tilde{m}_2$*, as requested.*

If ν *has bounded support, by the dominated convergence theorem, it can be easily checked that* $x(\tilde{m}) \to 0$ *when* $\tilde{m} \to \pm\infty$*.*

Denote $\tilde{m}_0$ *an edge of the domain of definition of* $x(\cdot)$*, to see the asymptotes as in Figure 2.5 we need to have the integral* $\int \frac{t\nu(dt)}{1+t\tilde{m}} \to \infty$ *diverge as* $\tilde{m} \to \tilde{m}_0$*. However, if* $\nu(dt)$ *behaves locally like* $(-1/\tilde{m}_0 - t)^\alpha$ *around* $t = -1/\tilde{m}_0$*, the integral* $\int \frac{t\nu(dt)}{1+t\tilde{m}}$ may not diverge *for a sufficiently large* α*, and therefore no such asymptote can be observed. Figure 8.7 provides an illustrating example of this peculiar behavior, here for edges of order* $\sqrt{-1/\tilde{m}_0 - t}$*.*

Exercise 9 (Alternative estimates of $\frac{1}{p}\operatorname{tr}(\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T})^2$)**.** *Let* $\mathbf{X} = \mathbf{C}^{\frac{1}{2}}\mathbf{Z}$ *for* $\mathbf{Z} \in \mathbb{R}^{p\times n}$ *with independent standard Gaussian entries, and* $\mathbf{C}$ *deterministic symmetric nonnegative definite, of bounded operator norm, and limiting spectral measure* ν*.*

Determine the limit, as $n, p \to \infty$ *and* $p/n \to c \in (0,\infty)$ *of the (empirical) second-order moment*

$$M_2 = \frac{1}{p}\operatorname{tr}\left(\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T}\right)^2$$

as a function of the moments of ν*.*

Retrieve the same result from the results of Exercise 1 along with the expression of the Stieltjes transform $m(z)$ *of the limiting spectrum* μ *of* $\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T}$*. It may be useful to first show that* $m(z)$ *is also solution to*

$$m(z) = \int \frac{\nu(dt)}{-z(1+ctm(z)) + (1-c)t}. \tag{8.10}$$

Correction 9 (Alternative estimates of $\frac{1}{p}\operatorname{tr}(\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T})^2$)**.** *A direct estimate is easily obtained by merely evaluating the expectation* $\mathbb{E}[\operatorname{tr}(\mathbf{C}^{\frac{1}{2}}\mathbf{Z}\mathbf{Z}^\mathsf{T}\mathbf{C}^{\frac{1}{2}}/n)^2]$*. By the unitary invariance of the Gaussian matrix* $\mathbf{Z}$ *(i.e.,* $\mathbf{U}\mathbf{Z} \sim \mathbf{Z}$ *in law for all orthogonal matrix* $\mathbf{U}$*), we may assume* $\mathbf{C} = \operatorname{diag}\{[\mathbf{C}]_{ii}\}_{i=1}^p$ *is diagonal. Therefore,*

$$\frac{1}{p}\mathbb{E}\left[\operatorname{tr}\left(\frac{1}{n}\mathbf{C}^{\frac{1}{2}}\mathbf{Z}\mathbf{Z}^\mathsf{T}\mathbf{C}^{\frac{1}{2}}\right)^2\right] = \frac{1}{n^2 p}\sum_{i,i'=1}^{p}\sum_{j,j'=1}^{n}\mathbf{C}_{ii}\mathbf{C}_{i'i'}\mathbb{E}[\mathbf{Z}_{ij}\mathbf{Z}_{i'j}\mathbf{Z}_{i'j'}\mathbf{Z}_{ij'}].$$

Of these sums, the case $i = i'$ *and* $j = j'$ *simultaneously brings* $\mathbb{E}[\mathbf{Z}_{ij}\mathbf{Z}_{i'j}\mathbf{Z}_{i'j'}\mathbf{Z}_{ij'}] = \mathbb{E}[\mathbf{Z}_{ij}^4] = 3$*; the other non-trivial case is when* $i = i'$ *with* $j \neq j'$*, or* $j = j'$ *with* $i \neq i'$*, which both yield* $\mathbb{E}[\mathbf{Z}_{ij}\mathbf{Z}_{i'j}\mathbf{Z}_{i'j'}\mathbf{Z}_{ij'}] = 1$*. As a consequence,*

$$\begin{aligned}\mathbb{E}\left[\frac{1}{p}\operatorname{tr}\left(\frac{1}{n}\mathbf{C}^{\frac{1}{2}}\mathbf{Z}\mathbf{Z}^\mathsf{T}\mathbf{C}^{\frac{1}{2}}\right)^2\right] &= \frac{3n\operatorname{tr}\mathbf{C}^2 + n(n-1)\operatorname{tr}\mathbf{C}^2 + n[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}\mathbf{C}^2]}{n^2 p} \\ &= \frac{1}{p}\operatorname{tr}\mathbf{C}^2 + c\left(\frac{1}{p}\operatorname{tr}\mathbf{C}\right)^2 + O(n^{-1})\end{aligned}$$

where we recall that $c = \lim p/n$*. In the large* n, p *limit, all empirical moments converge almost surely (this may be ensured by a control on their variances), and we thus find that*

$$M_2 = \frac{1}{p}\operatorname{tr}\mathbf{C}^2 + c\left(\frac{1}{p}\operatorname{tr}\mathbf{C}\right)^2 + O(n^{-1})$$

almost surely. It is in particular interesting to note that, as $c \to 0$*, we correctly recover the standard* $\operatorname{tr}\mathbf{C}^2/p$ *estimate. In the large dimensional regime though, the additional non-negligible term* $c(\operatorname{tr}\mathbf{C}/p)^2$ *contributes.*

We have, from Exercise 1 that

$$M_2 = \frac{1}{2}\frac{d^2\delta}{dz^2}(0) + o(1)$$

almost surely, where $\delta(z) = -\frac{1}{z}m(z^{-1}) = \int\frac{\mu(dt)}{1-tz}$*. Using* $m(z) = \frac{1}{c}\tilde{m}(z) + \frac{1-c}{cz}$ *and* $\tilde{m}(z) = (-z + c\int\frac{t\nu(dt)}{1+t\tilde{m}(z)})^{-1}$*, we may verify that*

$$m(z) = \int\frac{\nu(dt)}{-z(1+ctm(z)) + (1-c)t}.$$

From this expression, we find that

$$\delta(z) = -\frac{1}{z}m(z^{-1}) = \int\frac{\nu(dt)}{1 - ctz\delta(z) - (1-c)tz},$$

the derivative of which is given by

$$\delta'(z) = \int\frac{(1-c)t + ct\delta(z) + ctz\delta'(z)}{(1-ctz\delta(z) - (1-c)tz)^2}\,\nu(dt).$$

Remarking that $\delta(z) = \int \frac{\mu(dt)}{1-tz}$ *leads directly to* $\delta(0) = 1$ *and thus* $\delta'(0) = \int t\nu(dt)$*, as expected from Exercise 1. Differentiating a second time, we have*

$$\delta''(z) = \int \left[\frac{2ct\delta'(z) + ctz\delta''(z)}{(1 - ctz\delta(z) - (1-c)tz)^2} + 2\frac{((1-c)t + ct\delta(z) + ctz\delta'(z))^2}{(1 - ctz\delta(z) - (1-c)tz)^3} \right] \nu(dt)$$

which, after setting $z = 0$*, gives*

$$\delta''(0) = 2c\left(\int t\nu(dt)\right)^2 + 2\int t^2\nu(dt).$$

Consequently, we obtain

$$M_2 = c\left(\int t\nu(dt)\right)^2 + \int t^2\nu(dt) + o(1) = c\left(\frac{1}{p}\operatorname{tr}\mathbf{C}\right)^2 + \frac{1}{p}\operatorname{tr}\mathbf{C}^2 + o(1)$$

almost surely, as requested.

Exercise 10 (Location of the zeros of $\tilde{m}(z)$)**.** *Figure 2.7 and Remark 2.12 both show that the zeros* $\eta_1, \ldots, \eta_n$ *of* $m_{\mathbf{X}}(z)$*, the Stieltjes transform of a symmetric matrix* $\mathbf{X} \in \mathbb{R}^{n\times n}$*, are interlaced with the eigenvalues* $\lambda_1, \ldots, \lambda_n$ *of* $\mathbf{X}$*.*

In the sample covariance matrix case $\mathbf{X} = \frac{1}{n}\mathbf{Z}^\mathsf{T}\mathbf{C}\mathbf{Z}$ *with* $\mathbf{Z} \in \mathbb{R}^{p\times n}$ *having independent standard Gaussian entries and* $\mathbf{C}$ *with limited spectral measure* ν *of bounded and connected support, this means that (up to zero eigenvalues) the roots* η_i *of* $m_{\frac{1}{n}\mathbf{Z}^\mathsf{T}\mathbf{C}\mathbf{Z}}(z)$ *are all found in the limiting support of the empirical spectral measure* $\tilde{\mu}$ *of* $\frac{1}{n}\mathbf{Z}^\mathsf{T}\mathbf{C}\mathbf{Z}$, at the possible exception *of the leftmost* η_1*.*

Using a change of variable involving $\tilde{m}(z)$ *of the formula*

$$0 = \frac{1}{2\pi\imath}\oint_\Gamma \frac{dw}{w}$$

for all Γ *not enclosing zero, then the approximation* $\tilde{m}(z) = m_{\frac{1}{n}\mathbf{Z}^\mathsf{T}\mathbf{C}\mathbf{Z}}(z) + o(1)$ *and finally the residue theorem, show that no zero of* $m_{\frac{1}{n}\mathbf{Z}^\mathsf{T}\mathbf{C}\mathbf{Z}}(z)$ *can be found at macroscopic distance from the limiting support of* $\tilde{\mu}$*.*

This conclusion is of practical interest to statistical inference applications discussed in Section 2.4.1.3 and in particular to the explicit expression in (2.44) from (2.43), for which case this conclusion ensures a valid contour that circles around all *the* λ_i *poles and* η_i *poles, at least almost surely for sufficiently large* n, p*. (And the leftmost* η_1 *is* not *a problem.)*

Correction 10 (Location of the zeros of $\tilde{m}(z)$)**.** *Letting* $w = 1/\tilde{m}(z)$*, we find that, upon the validity of the change of variable for the contour* Γ*,*

$$\frac{1}{2\pi\imath}\oint_\Gamma \frac{dw}{w} = -\frac{1}{2\pi\imath}\oint_{\tilde{\Gamma}} \frac{\tilde{m}'(z)}{\tilde{m}(z)}\,dz.$$

The idea is then to take $\tilde{\Gamma}$ *any complex contour with leftmost real crossing slightly larger than zero and rightmost real crossing slightly smaller than the left edge of* *the support of $\tilde{\mu}$ (excluding the possible mass at 0 of course). From Figure 2.5 (and the discussions in that section), it appears that, for such a $\tilde{\Gamma}$, the associated Γ does not circle around zero, and thus the integral is null.*

As such, in the large n, p limit, we have

$$0 = \frac{1}{2\pi\imath} \oint_{\tilde{\Gamma}} \frac{\tilde{m}'_{\frac{1}{n}\mathbf{Z}^\mathsf{T}\mathbf{C}\mathbf{Z}}(z)}{\tilde{m}_{\frac{1}{n}\mathbf{Z}^\mathsf{T}\mathbf{C}\mathbf{Z}}(z)} dz + o(1)$$

almost surely. By residue calculus, the right-hand side integral equals the cardinality of the η_i's found inside $\tilde{\Gamma}$. As this is an integer and that, for all n, p large $o(1) < 1$, this cardinal must be zero: we therefore conclude that the smallest η_1 must also be found at an asymptotically vanishing distance from λ_1 (the smallest nonzero eigenvalue of $\frac{1}{n}\mathbf{Z}^\mathsf{T}\mathbf{C}\mathbf{Z}$) almost surely.

Exercise 11 (Additive spiked model)**.** *Similar to Theorem 2.13, show the phase transition threshold for the additive model $\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T} + \mathbf{P}$ for $\mathbf{X}$ having i.i.d. entries of zero mean, unit variance and low rank $\mathbf{P} = \sum_{i=1}^k \ell_i \mathbf{u}_i \mathbf{u}_i^\mathsf{T}$ with $\ell_1 > \ldots > \ell_k > 0$, is determined by the condition*

$$\ell_i > \sqrt{c}(1 + \sqrt{c})$$

with $c = \lim p/n$ as $p, n \to \infty$. Under this condition, show that the (almost sure) limiting value of the corresponding isolated eigenvalue $\hat{\lambda}_i$ of $\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T} + \mathbf{P}$ is given by

$$\hat{\lambda}_i \xrightarrow{a.s.} \lambda_i = 1 + \ell_i + \frac{c}{\ell_i - c}.$$

Further show, similar to Theorem 2.14 that, letting $\hat{\mathbf{u}}_i$ be the eigenvector associated with $\hat{\lambda}_i$, we have

$$|\hat{\mathbf{u}}_i^\mathsf{T} \mathbf{u}_i|^2 \xrightarrow{a.s.} 1 - \frac{c}{(\ell_i - c)^2}.$$

Correction 11 (Additive spiked model)**.** *The additive spiked model is in fact simpler to handle than the spiked covariance model in Theorem 2.13. We may answer the three questions (determination of the phase transition threshold, asymptotic spike position and eigenvector alignment) at once by writing*

$$|\hat{\mathbf{u}}_i^\mathsf{T} \mathbf{u}_i|^2 = -\frac{1}{2\pi\imath} \oint_{\Gamma_{\lambda_i}} \mathbf{u}_i^\mathsf{T} \left(\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T} + \mathbf{P} - z\mathbf{I}_p \right)^{-1} \mathbf{u}_i \, dz$$

for all large n, p almost surely, where Γ_{λ_i} positively surrounds λ_i only. By Woodbury's identity, Lemma 2.7, we obtain

$$|\hat{\mathbf{u}}_i^\mathsf{T} \mathbf{u}_i|^2 = -\frac{1}{2\pi\imath} \oint_{\Gamma_{\lambda_i}} \mathbf{u}_i^\mathsf{T} \mathbf{Q}(z) \mathbf{U}\mathbf{L}(\mathbf{I}_k + \mathbf{U}^\mathsf{T}\mathbf{Q}(z)\mathbf{U}\mathbf{L})^{-1}\mathbf{U}^\mathsf{T}\mathbf{Q}(z)\mathbf{u}_i \, dz \tag{8.11}$$

with $\mathbf{P} = \mathbf{U}\mathbf{L}\mathbf{U}^\mathsf{T}$ *and* $\mathbf{Q}(z) = (\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T} - z\mathbf{I}_p)^{-1}$ *the resolvent of* $\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T}$ *that has asymptotically no residue inside* Γ_{λ_i}. *Since* $\mathbf{U}^\mathsf{T}\mathbf{Q}(z)\mathbf{U} = m(z)\mathbf{I}_k + o_{\|\cdot\|}(1)$ *by Theorem 2.4, we find that,*

$$|\hat{\mathbf{u}}_i^\mathsf{T}\mathbf{u}_i|^2 \xrightarrow{a.s.} -\frac{1}{2\pi\imath}\oint_{\Gamma_{\lambda_i}} \frac{\ell_i m^2(z)}{1+\ell_i m(z)}\,dz. \tag{8.12}$$

By residue calculus, this term is null unless $1+\ell_i m(\lambda_i) = 0$*: this is the defining equation for* λ_i. *Since* $m(x)$ *increases on* $x \in ((1+\sqrt{c})^2, \infty)$ *with image* $(-1/(c+\sqrt{c}), 0)$*, this equation has a solution if and only if* $\ell_i > c + \sqrt{c}$*; this provides the phase transition threshold. Using again the definition of* $m(z)$*, we find in this case that* $\lambda_i = 1 + \ell_i + c/(\ell_i - c)$ *as requested.*

To characterize the (asymptotic) eigenvector alignment, exploiting again the expression

$$m'(z) = \frac{m^2(z)}{1 - \frac{cm^2(z)}{(1+cm(z))^2}} \tag{8.13}$$

the residue calculus finally gives, by l'Hôpital's rule (i.e., by Taylor expanding around λ_i *the numerator and denominator appearing in the residue calculus)*

$$|\hat{\mathbf{u}}_i^\mathsf{T}\mathbf{u}_i|^2 \xrightarrow{a.s.} \frac{m^2(\lambda_i)}{m'(\lambda_i)} = 1 - \frac{c}{(\ell_i - c)^2}$$

which concludes the proof.

Exercise 12 (Additive spiked model: the Wigner case)**.** *Let* $\mathbf{X}$ *be symmetric with* $[\mathbf{X}]_{ij}$, $i \geq j$, *i.i.d. with zero mean and unit variance. As in Exercise 11, show that the "spiked" phase transition threshold for the model* $\mathbf{X}/\sqrt{n} + \mathbf{P}$ *with* $\mathbf{P} = \sum_{i=1}^k \ell_i \mathbf{u}_i\mathbf{u}_i^\mathsf{T}$*, with* $\ell_1 > \ldots > \ell_k > 0$*, is determined by the condition*

$$\ell_i > 1$$

and that, under this condition, the isolated eigenvalue $\hat{\lambda}_i$ *of* $\frac{1}{\sqrt{n}}\mathbf{X} + \mathbf{P}$ *associated with* ℓ_i *satisfies*

$$\hat{\lambda}_i \xrightarrow{a.s.} \lambda_i = \ell_i + \frac{1}{\ell_i}. \tag{8.14}$$

Show finally that, for $\hat{\mathbf{u}}_i$ *the eigenvector associated with* $\hat{\lambda}_i$*, we have*

$$|\hat{\mathbf{u}}_i^\mathsf{T}\mathbf{u}_i|^2 \xrightarrow{a.s.} 1 - \frac{1}{\ell_i^2}. \tag{8.15}$$

Correction 12 (Additive spiked model: the Wigner case)**.** *The elements in the proof of Exercise 11 can be extensively reused, merely by replacing the definition of* $m(z)$ *by that of the Stieltjes transform of the semicircle law. In particular, here,* $m(z) = \frac{-1}{z+m(z)}$ *so that* $m'(z) = \frac{m^2(z)}{1-m^2(z)}$.

The asymptotic eigenvector alignment $|\hat{\mathbf{u}}_i^\mathsf{T}\mathbf{u}_i|^2$ *is again given by*

$$|\hat{\mathbf{u}}_i^\mathsf{T}\mathbf{u}_i|^2 \xrightarrow{a.s.} -\frac{1}{2\pi\imath}\oint_{\Gamma_{\lambda_i}} \frac{\ell_i m^2(z)}{1+\ell_i m(z)}\,dz \tag{8.16}$$

where the result is non-trivial only if $1+\ell_i m(\lambda_i) = 0$. *On the set* $(2,\infty)$ *(outside the support of the semicircle law),* $m(x)$ *is increasing with* $m(x \downarrow 2) = -1$. *The phase transition condition is thus* $\ell_i > 1$, *with* $\lambda_i = \ell_i + 1/\ell_i$, *as requested.*

The residue of the above integral is $\frac{m^2(\lambda_i)}{m'(\lambda_i)}$ *(again, as in Exercise 11) and we thus find that*

$$|\hat{\mathbf{u}}_i^\mathsf{T} \mathbf{u}_i|^2 \xrightarrow{a.s.} 1 - \frac{1}{\ell_i^2} \tag{8.17}$$

as requested.

Exercise 13 (Sketch of proof of Theorem 2.17)**.** *Inspired by the (sketch of) proof of Theorem 2.6, prove Theorem 2.17 using*

1. *the trace lemma adapted to Haar random matrices, Lemma 2.16; and*
2. *Stein's lemma adapted to Haar random matrices, Lemma 2.17.*

Correction 13 (Sketch of proof of Theorem 2.17)**.** *We first provide the derivation of Theorem 2.17 with the Haar trace lemma, Lemma 2.16. Before delving into the proof, we first proceed to a convenient change of variable: from the result of Theorem 2.17, for* $\mathbf{Q}(z) = (\frac{p}{n}\mathbf{C}^{\frac12}\mathbf{U}\mathbf{U}^\mathsf{T}\mathbf{C}^{\frac12} - z\mathbf{I}_p)^{-1}$ *with* $\mathbf{U} \in \mathbb{R}^{p\times n}$ *having* $n < p$ *columns from a* $p \times p$ *Haar random matrix, we have, for* $z < 0$, *the deterministic equivalent*

$$\mathbf{Q}(z) \leftrightarrow \bar{\mathbf{Q}}(z) = \left(\frac{p/n}{1+\delta(z)}\mathbf{C} - z\mathbf{I}_p \right)^{-1}$$

where $\delta(z)$ *is the unique positive solution to*

$$\delta(z) = \frac{p}{n} - 1 + \frac{p}{n}\frac{\delta(z)}{1+\delta(z)}\frac{1}{n}\operatorname{tr}\mathbf{C}\bar{\mathbf{Q}}(z).$$

This is indeed equivalent to the expected result by taking $\delta(z) = -\frac{p}{n}\frac{1}{z\tilde{m}_p(z)} - 1 > 0$ *for* $0 < -z\tilde{m}_p(z) < p/n \in (1,\infty)$. *We now proceed to the proof by following the steps of the proof of Theorem 2.6.*

For $\mathbf{u}_i \in \mathbb{R}^p$ *the* i*-th column of* $\mathbf{U}$, *we have*

$$\begin{aligned}\mathbf{I}_p + z\mathbb{E}[\mathbf{Q}] &= \mathbb{E}\left[\mathbf{Q}\cdot\frac{p}{n}\mathbf{C}^{\frac12}\mathbf{U}\mathbf{U}^\mathsf{T}\mathbf{C}^{\frac12}\right] = \frac{p}{n}\sum_{i=1}^n \mathbb{E}[\mathbf{Q}\mathbf{C}^{\frac12}\mathbf{u}_i\mathbf{u}_i^\mathsf{T}\mathbf{C}^{\frac12}] \\ &= \frac{p}{n}\sum_{i=1}^n \mathbb{E}\left[\frac{\mathbf{Q}_{-i}\mathbf{C}^{\frac12}\mathbf{u}_i\mathbf{u}_i^\mathsf{T}\mathbf{C}^{\frac12}}{1+\frac{p}{n}\mathbf{u}_i^\mathsf{T}\mathbf{C}^{\frac12}\mathbf{Q}_{-i}\mathbf{C}^{\frac12}\mathbf{u}_i}\right]\end{aligned}$$

where we denote $\mathbf{Q}_{-i} = (\frac{p}{n}\mathbf{C}^{\frac12}\mathbf{U}_{-i}\mathbf{U}_{-i}^\mathsf{T}\mathbf{C}^{\frac12} - z\mathbf{I}_p)^{-1}$ *for* $\mathbf{U}_{-i}\mathbf{U}_{-i}^\mathsf{T} = \mathbf{U}\mathbf{U}^\mathsf{T} - \mathbf{u}_i\mathbf{u}_i^\mathsf{T}$. *At this point, note that, unlike in the proof of Theorem 2.6, here* $\mathbf{Q}_{-i}$ *is no longer* independent *of* $\mathbf{u}_i$ *and the usual "trace lemma", Lemma 2.11, cannot be used. Instead, with Lemma 2.16, we have*

$$\begin{aligned}\frac{p}{n}\mathbb{E}[\mathbf{Q}_{-i}\mathbf{C}^{\frac12}\mathbf{u}_i\mathbf{u}_i\mathbf{C}^{\frac12}] &= \frac{p}{n}\frac{1}{p-n}\mathbb{E}[\mathbf{Q}_{-i}\mathbf{C}^{\frac12}(\mathbf{I}_p - \mathbf{U}_{-i}\mathbf{U}_{-i}^\mathsf{T})\mathbf{C}^{\frac12}] + o_{\|\cdot\|}(1) \\ &= \frac{1}{p-n}\mathbb{E}\left[\frac{p}{n}\mathbf{Q}_{-i}\mathbf{C} - (\mathbf{I}_p + z\mathbf{Q}_{-i})\right] + o_{\|\cdot\|}(1)\end{aligned}$$

as well as

$$\frac{p}{n}\mathbf{u}_i^\mathsf{T}\mathbf{C}^{\frac{1}{2}}\mathbf{Q}_{-i}\mathbf{C}^{\frac{1}{2}}\mathbf{u}_i = \frac{1}{p-n}\left(\frac{p}{n}\operatorname{tr}\mathbf{C}\mathbf{Q}_{-i} - (p + z\operatorname{tr}\mathbf{Q}_{-i})\right) + o(1)$$
$$\simeq \frac{1}{p-n}\left(\frac{p}{n}\operatorname{tr}\mathbf{C}\bar{\mathbf{Q}} - (p + z\operatorname{tr}\bar{\mathbf{Q}})\right) + o(1)$$

with $\frac{1}{p}\operatorname{tr}\mathbf{C}\mathbf{Q}_{-i} \simeq \frac{1}{p}\operatorname{tr}\mathbf{C}\mathbf{Q} \simeq \frac{1}{p}\operatorname{tr}\mathbf{C}\bar{\mathbf{Q}}$ *for some* $\bar{\mathbf{Q}} \leftrightarrow \mathbf{Q}$ *to be defined. Denoting*

$$\delta(z) \equiv \frac{p-n}{n} + \frac{p}{n}\frac{1}{n}\operatorname{tr}\mathbf{C}\bar{\mathbf{Q}} - \left(\frac{p}{n} + \frac{z}{n}\operatorname{tr}\bar{\mathbf{Q}}\right) \tag{8.18}$$

we thus have

$$\mathbf{I}_p + z\mathbb{E}[\mathbf{Q}] = \frac{\mathbb{E}[\mathbf{Q}]\left(\frac{p}{n}\mathbf{C} - z\mathbf{I}_p\right) - \mathbf{I}_p}{\delta(z)} + o_{\|\cdot\|}(1)$$

so that

$$\mathbb{E}[\mathbf{Q}(z)] \leftrightarrow \left(\frac{p/n}{1+\delta(z)}\mathbf{C} - z\mathbf{I}_p\right)^{-1} \equiv \bar{\mathbf{Q}}(z).$$

Plugging back into (8.18) *we obtain* $\delta(z) = \frac{p}{n} - 1 + \frac{p}{n}\frac{\delta(z)}{1+\delta(z)}\frac{1}{n}\operatorname{tr}\mathbf{C}\bar{\mathbf{Q}}(z)$ *which concludes the asymptotic approximation of the expectation* $\mathbb{E}[\mathbf{Q}] = \bar{\mathbf{Q}} + o_{\|\cdot\|}(1)$*. With additional concentration argument (for instance the Nash-Poincaré inequality for Haar matrices, Lemma 2.18) we conclude the proof.*

We now present the derivation of Theorem 2.17 with Stein's lemma for Haar matrices, Lemma 2.17, in the complex case for simplicity (see Remark 2.5). Since Lemma 2.17 only applies to square *Haar random matrices, we may rewrite the resolvent as*

$$\mathbf{Q}(z) = \left(\mathbf{C}^{\frac{1}{2}}\mathbf{U}\mathbf{D}\mathbf{U}^*\mathbf{C}^{\frac{1}{2}} - z\mathbf{I}_p\right)^{-1} \leftrightarrow \bar{\mathbf{Q}}(z)$$

for $\mathbf{U} \in \mathbb{C}^{p\times p}$ *a random Haar matrix and diagonal* $\mathbf{D} \in \mathbb{C}^{p\times p}$ *with* $\mathbf{D}_{ii} = d_i = \delta_{i\leq n}$ *and some* $\bar{\mathbf{Q}}(z)$ *to be specified. We start again from the relation* $\mathbb{E}[\mathbf{C}^{\frac{1}{2}}\mathbf{U}\mathbf{D}\mathbf{U}^*\mathbf{C}^{\frac{1}{2}}\mathbf{Q}] = \mathbf{I}_p + z\mathbb{E}[\mathbf{Q}]$*. Since* $\mathbf{U}$ *is unitary, it suffices to consider the case of diagonal* $\mathbf{C}$ *with* $[\mathbf{C}]_{ii} = \alpha_i$*. As such, defining*

$$f(\mathbf{U}) = \sqrt{\alpha}_q\mathbf{U}_{qk}d_l[\mathbf{U}^*]_{lm}\sqrt{\alpha}_m\mathbf{Q}_{mq} \in \mathbb{C}$$

we have

$$\frac{\partial\mathbf{U}^*}{\partial\mathbf{U}_{ij}} = -\mathbf{U}^*\mathbf{E}_{ij}\mathbf{U}^*$$
$$\frac{\partial\mathbf{Q}}{\partial\mathbf{U}_{ij}} = -\mathbf{Q}\mathbf{C}^{\frac{1}{2}}\mathbf{E}_{ij}\mathbf{D}\mathbf{U}^*\mathbf{C}^{\frac{1}{2}}\mathbf{Q} + \mathbf{Q}\mathbf{C}^{\frac{1}{2}}\mathbf{U}\mathbf{D}\mathbf{U}^*\mathbf{E}_{ij}\mathbf{U}^*\mathbf{C}^{\frac{1}{2}}\mathbf{Q}$$

and

$$\begin{aligned} f'_{ij}(\mathbf{U})\mathbf{U}_{ij'} &= \sqrt{\alpha_q}\delta_{qi}\delta_{kj}d_l[\mathbf{U}^*]_{lm}\sqrt{\alpha_m}\mathbf{Q}_{mq}\mathbf{U}_{ij'} \\ &- \sqrt{\alpha_q}\mathbf{U}_{qk}d_l[\mathbf{U}^*]_{li}[\mathbf{U}^*]_{jm}\sqrt{\alpha_m}\mathbf{Q}_{mq}\mathbf{U}_{ij'} \\ &- \sqrt{\alpha_q}\mathbf{U}_{qk}d_l[\mathbf{U}^*]_{lm}\sqrt{\alpha_m}[\mathbf{Q}\mathbf{C}^{\frac{1}{2}}]_{mi}[\mathbf{D}\mathbf{U}^*\mathbf{C}^{\frac{1}{2}}\mathbf{Q}]_{jq}\mathbf{U}_{ij'} \\ &+ \sqrt{\alpha_q}\mathbf{U}_{qk}d_l[\mathbf{U}^*]_{lm}\sqrt{\alpha_m}[\mathbf{Q}\mathbf{C}^{\frac{1}{2}}\mathbf{U}\mathbf{D}\mathbf{U}^*]_{mi}[\mathbf{U}^*\mathbf{C}^{\frac{1}{2}}\mathbf{Q}]_{jq}\mathbf{U}_{ij'} \end{aligned}$$

so that with Lemma 2.17 for $j = k$ and $j' = l$,

$$\begin{aligned} 0 = \mathbb{E}\left[\sum_{i=1}^p f'_{ij}(\mathbf{U})\mathbf{U}_{ij'}\right] &= \mathbb{E}[\sqrt{\alpha_q}\mathbf{U}_{ql}d_l[\mathbf{U}^*]_{lm}\sqrt{\alpha_m}\mathbf{Q}_{mq}] \\ &- \mathbb{E}[\sqrt{\alpha_q}\mathbf{U}_{qk}[\mathbf{U}^*]_{km}\sqrt{\alpha_m}\mathbf{Q}_{mq}d_l] \\ &- \mathbb{E}[\sqrt{\alpha_q}\mathbf{U}_{qk}[\mathbf{D}\mathbf{U}^*\mathbf{C}^{\frac{1}{2}}\mathbf{Q}]_{kq}d_l[\mathbf{U}^*]_{lm}\sqrt{\alpha_m}[\mathbf{Q}\mathbf{C}^{\frac{1}{2}}\mathbf{U}]_{ml}] \\ &+ \mathbb{E}[\sqrt{\alpha_q}\mathbf{U}_{qk}[\mathbf{U}^*\mathbf{C}^{\frac{1}{2}}\mathbf{Q}]_{kq}d_l[\mathbf{U}^*]_{lm}\sqrt{\alpha_m}[\mathbf{Q}\mathbf{C}^{\frac{1}{2}}\mathbf{U}\mathbf{D}]_{ml}] \end{aligned}$$

where we use $\mathbf{U}^*\mathbf{U} = \mathbf{I}_p$. Summing over k, l and m we get

$$\begin{aligned} 0 &= p\mathbb{E}[\alpha_q\mathbf{Q}_{qq}] - p\mathbb{E}[\alpha_q\mathbf{Q}_{qq}] - \mathbb{E}[[\mathbf{C}^{\frac{1}{2}}\mathbf{U}\mathbf{D}\mathbf{U}^*\mathbf{C}^{\frac{1}{2}}\mathbf{Q}]_{qq}\operatorname{tr}(\mathbf{Q}\mathbf{C})] \\ &+ \mathbb{E}[\alpha_q[\mathbf{Q}]_{qq}\operatorname{tr}(\mathbf{U}^*\mathbf{C}^{\frac{1}{2}}\mathbf{Q}\mathbf{C}^{\frac{1}{2}}\mathbf{U}\mathbf{D})] \end{aligned}$$

and with a concentration argument for the traces we have

$$\begin{aligned} \mathbb{E}[\mathbf{Q}]_{qq} &= \frac{\operatorname{tr}(\mathbb{E}[\mathbf{Q}]\mathbf{C})}{\alpha_q\mathbb{E}[\operatorname{tr}(\mathbf{C}^{\frac{1}{2}}\mathbf{U}\mathbf{D}\mathbf{U}^*\mathbf{C}^{\frac{1}{2}}\mathbf{Q})] - z\operatorname{tr}(\mathbb{E}[\mathbf{Q}]\mathbf{C})} + o(1) \\ &= \frac{\frac{1}{p}\operatorname{tr}\bar{\mathbf{Q}}\mathbf{C}}{\alpha_q + \alpha_q\frac{z}{p}\operatorname{tr}\bar{\mathbf{Q}} - \frac{z}{p}\operatorname{tr}\bar{\mathbf{Q}}\mathbf{C}} + o(1) \end{aligned}$$

so that $\bar{\mathbf{Q}}$ should take the form

$$\bar{\mathbf{Q}} = \left(\frac{p + z\operatorname{tr}\bar{\mathbf{Q}}}{\operatorname{tr}\mathbf{C}\bar{\mathbf{Q}}}\mathbf{C} - z\mathbf{I}_p\right)^{-1}$$

and therefore

$$\frac{1}{p}\operatorname{tr}\mathbf{C}\bar{\mathbf{Q}} = \frac{1}{p}\operatorname{tr}\mathbf{C}\left(\frac{1 + \frac{z}{p}\operatorname{tr}\bar{\mathbf{Q}}}{\frac{1}{p}\operatorname{tr}\mathbf{C}\bar{\mathbf{Q}}}\mathbf{C} - z\mathbf{I}_p\right)^{-1} + o(1). \tag{8.19}$$

Similarly, for $\tilde{\mathbf{Q}}(z) = (\mathbf{D}^{\frac{1}{2}}\mathbf{U}^*\mathbf{C}\mathbf{U}\mathbf{D}^{\frac{1}{2}} - z\mathbf{I}_n)^{-1}$ we have

$$\frac{1}{p}\operatorname{tr}\mathbf{D}\bar{\tilde{\mathbf{Q}}} = \frac{1}{p}\operatorname{tr}\mathbf{D}\left(\frac{1 + \frac{z}{p}\operatorname{tr}\bar{\mathbf{Q}}}{\frac{1}{p}\operatorname{tr}\mathbf{D}\bar{\tilde{\mathbf{Q}}}}\mathbf{D} - z\mathbf{I}_n\right)^{-1} + o(1). \tag{8.20}$$

To connect (8.19) to (8.20) and close the loop, it suffices to note that

$$\frac{1}{p}\operatorname{tr}\mathbf{C}\bar{\mathbf{Q}}\cdot\frac{1}{p}\operatorname{tr}\mathbf{D}\bar{\tilde{\mathbf{Q}}} = \left(1+\frac{z}{p}\operatorname{tr}\bar{\mathbf{Q}}\right)\cdot\frac{1}{p}\operatorname{tr}\bar{\mathbf{Q}} + o(1).$$

where we use the fact that $\mathbf{D}^{\frac{1}{2}}\mathbf{U}^*\mathbf{C}^{\frac{1}{2}}\mathbf{Q} = \tilde{\mathbf{Q}}\mathbf{D}^{\frac{1}{2}}\mathbf{U}^*\mathbf{C}^{\frac{1}{2}}$*. As a consequence, denoting* $\tilde{m}_p(z) = -\frac{1}{z}\frac{p+z\operatorname{tr}\bar{\mathbf{Q}}}{\operatorname{tr}\mathbf{C}\bar{\mathbf{Q}}}$ *we get*

$$\tilde{m}_p(z) = \left(-z-(zc+z^2\tilde{m}_p(z))\frac{1}{p}\operatorname{tr}\mathbf{C}\bar{\mathbf{Q}}\right)^{-1}, \quad \bar{\mathbf{Q}}(z) = -\frac{1}{z}\left(\mathbf{I}_p+\tilde{m}_p(z)\mathbf{C}\right)^{-1}$$

which, together with additional concentration arguments, concludes the proof.

Exercise 14 (Higher-order deterministic equivalent)**.** *Theorem 2.4 provides a deterministic equivalent for the resolvent* $\mathbf{Q} = \left(\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T} - z\mathbf{I}_p\right)^{-1}$ *for* $\mathbf{X}\in\mathbb{R}^{p\times n}$ *having i.i.d. zero mean and unit variance entries, which, according to Notation 1, provides access to the asymptotic behavior of* $\mathbf{a}^\mathsf{T}\mathbf{Q}\mathbf{b}$*. In many machine learning applications, however, the object of natural interest (e.g., the mean squared error in a regression context and the variance in a classification context) often involves the asymptotic behavior of* $\mathbf{a}^\mathsf{T}\mathbf{Q}\mathbf{A}\mathbf{Q}\mathbf{b}$ *which requires a deterministic equivalent for random matrices of the type* $\mathbf{Q}\mathbf{A}\mathbf{Q}$*, for some* $\mathbf{A}$ *independent of* $\mathbf{Q}$*. In particular, for* $\mathbf{Q}\leftrightarrow\bar{\mathbf{Q}}$ *(such that* $\|\mathbb{E}[\mathbf{Q}]-\bar{\mathbf{Q}}\|\to 0$*),* $\bar{\mathbf{Q}}\mathbf{A}\bar{\mathbf{Q}}$ *is in general* not *a deterministic equivalent for* $\mathbf{Q}\mathbf{A}\mathbf{Q}$*. This is due to the fact that*

$$\mathbb{E}[\mathbf{Q}\mathbf{A}\mathbf{Q}] \not\simeq \mathbb{E}[\mathbf{Q}]\mathbf{A}\mathbb{E}[\mathbf{Q}].$$

Instead, prove that, in the setting of Theorem 2.4, one has

$$\mathbf{Q}(z)\mathbf{A}\mathbf{Q}(z) \leftrightarrow m^2(z)\mathbf{A} + \frac{1}{n}\operatorname{tr}\mathbf{A}\cdot\frac{m'(z)m^2(z)}{(1+cm(z))^2}\mathbf{I}_p. \tag{8.21}$$

As a sanity check, using the fact that $\partial\mathbf{Q}(z)/\partial z = \mathbf{Q}^2(z)$ *and taking* $\mathbf{A}=\mathbf{I}_p$ *in the equation above, confirm that*

$$\mathbf{Q}^2(z)\leftrightarrow m'(z)\mathbf{I}_p$$

for $m'(z) = \frac{m^2(z)}{1-\frac{cm^2(z)}{(1+cm(z))^2}}$ *obtained from differentiating the Marčenko–Pastur equation (2.9).*

Correction 14 (Higher-order deterministic equivalent)**.** *First recall from Theorem 2.4 that*

$$\mathbf{Q}(z)\leftrightarrow\bar{\mathbf{Q}}(z) = m(z)\mathbf{I}_p = \left(\frac{1}{1+cm(z)} - z\right)^{-1}\mathbf{I}_p, \quad m(z) = \frac{1}{-z+\frac{1}{1+cm(z)}}.$$

We then have

$$
\begin{aligned}
\mathbb{E}[\mathbf{QAQ}] &= \mathbb{E}[\mathbf{QA}\bar{\mathbf{Q}}] + \mathbb{E}[\mathbf{QA}(\mathbf{Q} - \bar{\mathbf{Q}})] \\
&= \bar{\mathbf{Q}}\mathbf{A}\bar{\mathbf{Q}} + \mathbb{E}\left[\mathbf{QAQ}\left(\frac{\mathbf{I}_p}{1 + \frac{1}{n}\operatorname{tr}\bar{\mathbf{Q}}} - \frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T}\right)\right]\bar{\mathbf{Q}} + o_{\|\cdot\|}(1) \\
&= \bar{\mathbf{Q}}\mathbf{A}\bar{\mathbf{Q}} + \frac{\mathbb{E}[\mathbf{QAQ}]\bar{\mathbf{Q}}}{1 + \frac{1}{n}\operatorname{tr}\bar{\mathbf{Q}}} - \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\frac{\mathbf{QAQ}_{-i}\mathbf{x}_i\mathbf{x}_i^\mathsf{T}}{1 + \frac{1}{n}\mathbf{x}_i^\mathsf{T}\mathbf{Q}_{-i}\mathbf{x}_i}\right]\bar{\mathbf{Q}} + o_{\|\cdot\|}(1) \\
&= \bar{\mathbf{Q}}\mathbf{A}\bar{\mathbf{Q}} + \frac{\mathbb{E}[\mathbf{QAQ}]\bar{\mathbf{Q}}}{1 + \frac{1}{n}\operatorname{tr}\bar{\mathbf{Q}}} - \frac{1}{n}\sum_{i=1}^n \frac{\mathbb{E}[\mathbf{Q}_{-i}\mathbf{AQ}_{-i}\mathbf{x}_i\mathbf{x}_i^\mathsf{T}]}{1 + \frac{1}{n}\operatorname{tr}\bar{\mathbf{Q}}}\bar{\mathbf{Q}} \\
&+ \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\frac{\mathbf{Q}_{-i}\mathbf{x}_i\frac{1}{n}\mathbf{x}_i^\mathsf{T}\mathbf{Q}_{-i}\mathbf{AQ}_{-i}\mathbf{x}_i\mathbf{x}_i^\mathsf{T}}{(1 + \frac{1}{n}\operatorname{tr}\bar{\mathbf{Q}})^2}\right]\bar{\mathbf{Q}} + o_{\|\cdot\|}(1) \\
&= \bar{\mathbf{Q}}\mathbf{A}\bar{\mathbf{Q}} + \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\mathbf{Q}_{-i}\frac{\frac{1}{n}\operatorname{tr}(\mathbf{Q}_{-i}\mathbf{AQ}_{-i})}{(1 + \frac{1}{n}\operatorname{tr}\bar{\mathbf{Q}})^2}\right]\bar{\mathbf{Q}} \\
&= \bar{\mathbf{Q}}\mathbf{A}\bar{\mathbf{Q}} + \frac{\frac{1}{n}\operatorname{tr}(\mathbb{E}[\mathbf{QAQ}])}{(1 + \frac{1}{n}\operatorname{tr}\bar{\mathbf{Q}})^2}\bar{\mathbf{Q}}^2 + o_{\|\cdot\|}(1)
\end{aligned}
$$

where we apply Lemma 2.8 twice and use the fact that $\mathbb{E}[\mathbf{Q}] = \bar{\mathbf{Q}} + o_{\|\cdot\|}(1)$ *and* $\mathbb{E}[\mathbf{QAQ}] = \mathbb{E}[\mathbf{Q}_{-i}\mathbf{AQ}_{-i}] + o_{\|\cdot\|}(1)$ *for* $\mathbf{A}$ *of bounded norm. Taking the (normalized) trace and gathering the terms proportional to* $\operatorname{tr}(\mathbb{E}[\mathbf{QAQ}])$*, we obtain*

$$
\begin{aligned}
\frac{1}{n}\operatorname{tr}(\mathbb{E}[\mathbf{QAQ}]) &= \left(1 - \frac{\frac{1}{n}\operatorname{tr}\bar{\mathbf{Q}}^2}{(1 + \frac{1}{n}\operatorname{tr}\bar{\mathbf{Q}})^2}\right)^{-1}\frac{1}{n}\operatorname{tr}\bar{\mathbf{Q}}\mathbf{A}\bar{\mathbf{Q}} = \frac{m^2(z)\frac{1}{n}\operatorname{tr}\mathbf{A}}{1 - \frac{cm^2(z)}{(1+cm(z))^2}} + o(1) \\
&= m'(z)\frac{1}{n}\operatorname{tr}\mathbf{A} + o(1)
\end{aligned}
$$

where we recall that $m'(z) = \frac{m^2(z)}{1 - \frac{cm^2(z)}{(1+cm(z))^2}}$ *by differentiating (2.9) and*

$$
\mathbf{QAQ} \leftrightarrow \bar{\mathbf{Q}}\mathbf{A}\bar{\mathbf{Q}} + \frac{1}{n}\operatorname{tr}\mathbf{A}\cdot\frac{m'(z)}{(1+cm(z))^2}\bar{\mathbf{Q}}^2 = m^2(z)\mathbf{A} + \frac{1}{n}\operatorname{tr}\mathbf{A}\cdot\frac{m'(z)m^2(z)}{(1+cm(z))^2}\mathbf{I}_p.
$$

In particular, with $\mathbf{A} = \mathbf{I}_p$*, we obtain* $\mathbf{Q}(z)\mathbf{AQ}(z) \leftrightarrow m'(z)\mathbf{I}_p$*. It is however worthy noting that. for low rank* $\mathbf{A}$*, say* $\operatorname{rank}(\mathbf{A}) = k$ *fixed as* $n, p \to \infty$*, then one obtain follow the derivation above that*

$$
\mathbf{QAQ} \leftrightarrow \bar{\mathbf{Q}}\mathbf{A}\bar{\mathbf{Q}} = m^2(z)\mathbf{A} \tag{8.22}
$$

since the additional term proportional to $\frac{1}{n}\operatorname{tr}\mathbf{A}$ *is negligible according to Lemma 2.9.*

Exercise 15 (Concentration of matrix quadratic forms)**.** *Recalling the definitions and notations of Section 2.7, let* $\mathbf{X} \in \mathbb{R}^{p\times n}$ *be a random matrix satisfying*

$$
\mathbf{X} \propto Ce^{c\cdot^2}, \text{ and } \|\mathbb{E}[\mathbf{X}]\| \le K
$$

for some $K, C, c > 0$. Given $\mathbf{A} \in \mathbb{R}^{p\times p}$ deterministic, we aim to prove the linear concentration of $\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X}$ in $(\mathbb{R}^{n\times n}, \|\cdot\|_F)$. To this end, we consider a deterministic matrix $\mathbf{B} \in \mathbb{R}^{n\times n}$ such that $\|\mathbf{B}\|_F \leq 1$ and study the behavior of $\operatorname{tr}(\mathbf{B}\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X})$. *Consider first the singular value decomposition*

$$\mathbf{A} = \mathbf{U_A}\mathbf{\Lambda_A}\mathbf{V_A}^\mathsf{T}, \quad \mathbf{B} = \mathbf{U_B}\mathbf{\Lambda_B}\mathbf{V_B}^\mathsf{T},$$

with $\mathbf{U_A}, \mathbf{V_A} \in \mathbb{R}^{p\times p}$ and $\mathbf{U_B}, \mathbf{V_B} \in \mathbb{R}^{n\times n}$ orthogonal matrices, $\mathbf{\Lambda_A} \in \mathbb{R}^{p\times p}$, $\mathbf{\Lambda_B} \in \mathbb{R}^{n\times n}$ diagonal matrices, and define $\tilde{\mathbf{X}}_1 = \mathbf{U_A}^\mathsf{T}\mathbf{X}\mathbf{V_B}, \tilde{\mathbf{X}}_2 = \mathbf{V_A}^\mathsf{T}\mathbf{X}\mathbf{U_B} \in \mathbb{R}^{p\times n}$. In the sequel, the constants $K', C', c' > 0$ are understood only depending on K, C, c and might change from line to line.

First show that there exist $K', C', c' > 0$ such that, for $t > K'\sqrt{\log(np)}$ and $\tilde{\mathbf{X}} \in \{\tilde{\mathbf{X}}_1, \tilde{\mathbf{X}}_2\}$

$$\mathbb{P}\left(\|\tilde{\mathbf{X}} - \mathbb{E}[\tilde{\mathbf{X}}]\|_\infty \geq t\right) \leq C'e^{-c't^2/\log(np)}.$$

Deduce from the bound $\|\mathbb{E}[\mathbf{X}]\| \leq K$ that there exists a constant $K' > 0$ depending only on K, C, c such that

$$\mathbb{E}[\|\tilde{\mathbf{X}}\|_\infty] \leq K'\sqrt{\log(np)}.$$

This established, introduce the set $\mathcal{A}_\theta = \{\mathbf{X} \in \mathbb{R}^{p\times n}, \max\{\|\tilde{\mathbf{X}}_1\|_\infty, \|\tilde{\mathbf{X}}_2\|_\infty\} \leq \theta\} \subset \mathbb{R}^{p\times n}$ and show that for all $\theta \geq K'\sqrt{\log(np)}$ with $K' > 1$, we have

$$\mathbb{P}(\mathbf{X} \in \mathcal{A}_\theta^c) \leq C'e^{-c'\theta^2}$$

and that the mapping $\mathbf{X} \mapsto \operatorname{tr}(\mathbf{B}\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X})$ is $\theta\|\mathbf{A}\|_F$-Lipschitz on $\mathcal{A}_\theta$.

Introduce M, a median of $\operatorname{tr}(\mathbf{B}\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X})$, *and note that*

$$\mathbb{P}\left(\left|\operatorname{tr}(\mathbf{B}\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X}) - M\right| \geq t, \mathbf{X} \in \mathcal{A}_\theta\right) \leq C'e^{-c't^2/(\theta\|\mathbf{A}\|_F)^2}.$$

Conclude by carefully choosing the parameter $\theta \geq K'\sqrt{\log(np)}$ and showing that

$$\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X} \in \mathbb{E}[\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X}] \pm C'e^{-c'\cdot^2/(\log(np)\|\mathbf{A}\|_F^2)} + C'e^{-c'\cdot/\|\mathbf{A}\|_F}.$$

Correction 15 (Concentration of matrix quadratic forms)**.** *First note that for $\|\mathbf{U_A}\| = \|\mathbf{U_B}\| = \|\mathbf{V_A}\| = \|\mathbf{V_B}\| = 1$, it follows from $\mathbf{X} \propto Ce^{c\cdot^2}$ that both $\tilde{\mathbf{X}}_1 = \mathbf{U_A}^\mathsf{T}\mathbf{X}\mathbf{V_B} \propto Ce^{c\cdot^2}$ and $\tilde{\mathbf{X}}_2 = \mathbf{V_A}^\mathsf{T}\mathbf{X}\mathbf{U_B} \propto Ce^{c\cdot^2}$ (linear and thus Lipschitz with constant one). Denote $\mathbf{e}_i \in \mathbb{R}^n$ the canonical basis vector of $\mathbb{R}^n$ such that $[\mathbf{e}_i]_j = \delta_{ij}$, we thus have for $\mathbf{X} \in \{\tilde{\mathbf{X}}_1, \tilde{\mathbf{X}}_2\}$*

$$\mathbb{P}\left(\|\tilde{\mathbf{X}} - \mathbb{E}[\tilde{\mathbf{X}}]\|_\infty \geq t\right) \leq \mathbb{P}\left(\sup_{\substack{1\leq i\leq p\\1\leq j\leq n}} \left|\mathbf{e}_i^\mathsf{T}(\tilde{\mathbf{X}} - \mathbb{E}[\tilde{\mathbf{X}}])\mathbf{e}_j\right| \geq t\right) \leq np \cdot 2Ce^{-ct^2}.$$

Since $Ce^{\log(np)}e^{-ct^2} \leq C'e^{-c't^2/\log(np)}$ for $t \geq K'\sqrt{\log(np)}$ for some constants K', C', c' depending only on K, C, c, we have

$$\mathbb{P}\left(\|\tilde{\mathbf{X}} - \mathbb{E}[\tilde{\mathbf{X}}]\|_\infty \geq t\right) \leq C'e^{-c't^2/\log(np)}$$

for $t \geq K'\sqrt{\log(np)}$ *and therefore*

$$\begin{aligned}
\mathbb{E}[\|\tilde{\mathbf{X}}\|_\infty] &= \mathbb{E}\left[\|\mathbb{E}[\tilde{\mathbf{X}}] + (\tilde{\mathbf{X}} - \mathbb{E}[\tilde{\mathbf{X}}])\|_\infty\right] \\
&\leq \|\mathbb{E}[\tilde{\mathbf{X}}]\|_\infty + \mathbb{E}[\|\tilde{\mathbf{X}} - \mathbb{E}[\tilde{\mathbf{X}}]\|_\infty] \\
&\leq \|\mathbb{E}[\tilde{\mathbf{X}}]\| + \int_0^\infty \mathbb{P}\left(\|\tilde{\mathbf{X}} - \mathbb{E}[\tilde{\mathbf{X}}]\|_\infty \geq t\right) dt \\
&\leq K + K'\sqrt{\log(np)} + \int_{K'\log(np)}^\infty \mathbb{P}\left(\|\tilde{\mathbf{X}} - \mathbb{E}[\tilde{\mathbf{X}}]\|_\infty \geq t\right) dt \\
&\leq K'\sqrt{\log(np)} + C'\sqrt{\log(np)} \int_0^\infty e^{-c't^2} dt \leq K'\sqrt{\log(np)}.
\end{aligned}$$

Next, note that for $\theta \geq 2K'\sqrt{\log(np)}$*, we have* $\theta \geq \frac{\theta}{2} + \mathbb{E}[\|\tilde{\mathbf{X}}\|_\infty]$*. Since* $\tilde{\mathbf{X}} \propto Ce^{-c\cdot^2}$*, we have* $\|\tilde{\mathbf{X}}\|_\infty = \sup_{ij} |\mathbf{e}_i^\mathsf{T}\tilde{\mathbf{X}}\mathbf{e}_j| \propto Ce^{-c\cdot^2}$ *and*

$$\begin{aligned}
\mathbb{P}(\mathbf{X} \in \mathcal{A}_\theta^c) &\leq \mathbb{P}\left(\|\tilde{\mathbf{X}}\|_\infty \geq \theta\right) \leq \mathbb{P}\left(\left|\|\tilde{\mathbf{X}}\|_\infty - \mathbb{E}[\|\tilde{\mathbf{X}}\|_\infty]\right| \geq \frac{\theta}{2}\right) \\
&\leq np \cdot 2Ce^{-c\theta^2/4} \leq C'e^{-c'\theta^2}
\end{aligned}$$

for $\mathcal{A}_\theta = \{\mathbf{X} \in \mathbb{R}^{p\times n}, \max\{\|\tilde{\mathbf{X}}_1\|_\infty, \|\tilde{\mathbf{X}}_2\|_\infty\} \leq \theta\}$ *and* $K' > 1$*. Besides, with* $\mathbf{\Lambda_A} = \operatorname{diag}\{\lambda_i(\mathbf{A})\}_{i=1}^p$ *and* $\mathbf{\Lambda_B} = \operatorname{diag}\{\lambda_i(\mathbf{B})\}_{i=1}^n$*, we find that for any* $\mathbf{X} \in \mathcal{A}_\theta$*,*

$$\operatorname{tr}(\mathbf{B}\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X}) = \operatorname{tr}(\mathbf{\Lambda_B}\tilde{\mathbf{X}}_1^\mathsf{T}\mathbf{\Lambda_A}\tilde{\mathbf{X}}_2) = \operatorname{diag}(\mathbf{\Lambda_A})^\mathsf{T}(\tilde{\mathbf{X}}_1 \odot \tilde{\mathbf{X}}_2)\operatorname{diag}(\mathbf{\Lambda_B})$$

where $\odot$ *is the Hadamard entry-wise product. Therefore*

$$|\operatorname{tr}(\mathbf{B}\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X})| \leq \|\mathbf{A}\|_F \cdot \|\mathbf{B}\|_F \cdot \|\tilde{\mathbf{X}}\|_F \cdot \|\tilde{\mathbf{X}}\|_\infty \leq \theta\|\mathbf{A}\|_F \cdot \|\mathbf{X}\|_F$$

which means that, for $\|\mathbf{B}\|_F \leq 1$*, the map* $\mathbf{X} \mapsto \operatorname{tr}(\mathbf{B}\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X})$ *is* $\theta\|\mathbf{A}\|_F$*-Lipschitz on* $\mathcal{A}_\theta$*. As a consequence, we can bound*

$$\begin{aligned}
&\mathbb{P}\left(\left|\operatorname{tr}(\mathbf{B}\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X}) - \mathbb{E}[\operatorname{tr}(\mathbf{B}\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X})]\right| \geq t\right) \\
&\leq \mathbb{P}\left(\left|\operatorname{tr}(\mathbf{B}\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X}) - M\right| \geq t, \mathbf{X} \in \mathcal{A}_\theta\right) + \mathbb{P}\left(\mathbf{X} \in \mathcal{A}_\theta^c\right) \\
&\leq C'e^{-c't^2/(\theta\|\mathbf{A}\|_F)^2} + C'e^{-c'\theta^2}
\end{aligned}$$

where we used, in the last inequality, the Lipschitz nature of the map $\mathbf{X} \mapsto \operatorname{tr}(\mathbf{B}\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X})$ *on the set* $\mathcal{A}_\theta$*. Finally, by choosing* $\theta = \max(K'\sqrt{\log(np)}, \sqrt{t/\|\mathbf{A}\|_F})$*, we can bound*

$$\begin{aligned}
&\mathbb{P}\left(\left|\operatorname{tr}(\mathbf{B}\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X}) - \mathbb{E}[\operatorname{tr}(\mathbf{B}\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X})]\right| \geq t\right) \\
&\leq C'e^{-c't^2/\left(K'\sqrt{\log(pn)}\|\mathbf{A}\|_F\right)^2} + C'e^{-c't/\|\mathbf{A}\|_F},
\end{aligned}$$

from which we conclude on the linear concentration of $\mathbf{X}^\mathsf{T}\mathbf{A}\mathbf{X}$*.*

Exercise 16 (Towards spiked models in random tensors)**.** *Let $\mathcal{Y} \in \mathbb{R}^{n\times n\times n}$ be a three-way symmetric tensor, i.e., such that $[\mathcal{Y}]_{ijk}$ is constant to exchanges of its indexes, defined by*

$$\mathcal{Y} = \ell \mathbf{x} \otimes \mathbf{x} \otimes \mathbf{x} + \frac{1}{\sqrt{n}} \mathcal{W}$$

where $\mathcal{W} \in \mathbb{R}^{n\times n\times n}$ has independent $\mathcal{N}(0,1)$ entries up to symmetry, deterministic $\mathbf{x} \in \mathbb{R}^n$ of unit norm, and $[\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}]_{ijk} = a_i b_j c_k$.

A possible definition of the "eigenvalue-eigenvector" pair $(\hat{\lambda}, \hat{\mathbf{u}})$ (without loss of generality such that $\hat{\lambda} \geq 0$ and $\|\hat{\mathbf{u}}\| = 1$) of a symmetric tensor $\mathcal{Y}$ is the solution to [Lim, 2005]

$$\mathcal{Y} \cdot \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = \hat{\lambda} \hat{\mathbf{u}} \tag{8.23}$$

where $\mathcal{A} \cdot \mathbf{a} \cdot \mathbf{b} = \sum_{ij} [\mathcal{A}]_{ij\cdot} a_i b_j \in \mathbb{R}^n$ is the contraction of the tensor $\mathcal{A}$ on the vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. The objective here is to characterize the (possible) spike $\hat{\lambda}$ as well as the associated eigenvector alignment $|\hat{\mathbf{u}}^\mathsf{T} \mathbf{x}|$ between the dominant eigenvector and $\mathbf{x}$.

Show first that the matrix $\mathbf{Y}_\mathbf{x} = \mathcal{Y} \cdot \mathbf{x} = \sum_{i=1}^n \mathcal{Y}_{i\cdot\cdot} x_i \in \mathbb{R}^{n\times n}$ takes the form

$$\mathbf{Y}_\mathbf{x} = \ell \mathbf{x}\mathbf{x}^\mathsf{T} + \frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \mathbf{W}_i \tag{8.24}$$

where $\mathbf{W}_i \in \mathbb{R}^{n\times n}$ is the i-th "layer" matrix of the tensor $\mathcal{W}$ such that $[\mathbf{W}_i]_{ab} = \mathcal{W}_{iab}$.

Using the Gaussian method discussed in Section 2.2.2.2, show that the limiting spectral measure of $\mathbf{Y}_\mathbf{x}$ is the semicircle law supported on $[-2, 2]$ (we may discard the rank-one matrix $\ell \mathbf{x}\mathbf{x}^\mathsf{T}$ to retrieve this result). Then, using a spiked model analysis as in Section 2.5, show that

- *for all $\ell > 0$, there must exist an isolated eigenvalue $\hat{\lambda}_\mathbf{x}$ of $\mathbf{Y}_\mathbf{x}$ (thus no phase transition) asymptotically equal to (with high probability)*

$$\hat{\lambda}_\mathbf{x} \to \lambda_\mathbf{x} = \sqrt{\ell^2 + 4};$$

- *the eigenvector $\hat{\mathbf{u}}_\mathbf{x}$ associated with $\hat{\lambda}_\mathbf{x}$ satisfies (again with high probability)*

$$|\hat{\mathbf{u}}_\mathbf{x}^\mathsf{T} \mathbf{x}|^2 \to \frac{\ell}{\sqrt{\ell^2 + 4}}.$$

Conclude on an asymptotic upper bound for the quantity $\hat{\lambda} |\hat{\mathbf{u}}^\mathsf{T} \mathbf{x}|$.

Correction 16 (Towards spiked models in random tensors)**.** *After establishing (8.24), the main technical difficulty to obtain the semicircle limit lies in handling the multi-way symmetry of the tensor $\mathcal{W}$. Denoting the resolvent*

$\mathbf{Q} = (\sum_{i=1}^n x_i \mathbf{W}_i/\sqrt{n} - z\mathbf{I}_n)^{-1}$ *of* $\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \mathbf{W}_i$*, we write, with the help of Stein's lemma, Lemma 2.13, that*

$$\frac{1}{\sqrt{n}} \sum_{l=1}^n x_l \mathbb{E}[\mathbf{W}_l \mathbf{Q}]_{ij} = \frac{1}{\sqrt{n}} \sum_{m,l} x_l \mathbb{E}[\mathcal{W}_{lim}[\mathbf{Q}]_{mj}] = \frac{1}{\sqrt{n}} \sum_{m,l} x_l \mathbb{E}\left[\frac{\partial [\mathbf{Q}]_{mj}}{\partial \mathcal{W}_{lim}}\right]. \tag{8.25}$$

At this point, note that by symmetry $\mathcal{W}_{abc} = \mathcal{W}_{bac} = \mathcal{W}_{cab} = \mathcal{W}_{cba} = \mathcal{W}_{bca} = \mathcal{W}_{acb}$ *and therefore*

$$\begin{aligned} \frac{\partial \mathbf{Q}}{\partial \mathcal{W}_{abc}} &= -\frac{1}{\sqrt{n}} \sum_{k=1}^n x_k \mathbf{Q} \frac{\partial \mathbf{W}_k}{\partial \mathcal{W}_{abc}} \mathbf{Q} \\ &= -\frac{1}{\sqrt{n}} \sum_{k=1}^n x_k \mathbf{Q} \left[(\mathbf{E}_{ab} + \mathbf{E}_{ba})\delta_{kc} + (\mathbf{E}_{ca} + \mathbf{E}_{ac})\delta_{kb} + (\mathbf{E}_{bc} + \mathbf{E}_{cb})\delta_{ka}\right] \mathbf{Q} \end{aligned}$$

with $\mathbf{E}_{ab} \in \mathbb{R}^{n \times n}$ *the indicator matrix such that* $[\mathbf{E}_{ab}]_{a'b'} = \delta_{aa'}\delta_{bb'}$*. Plugging the above expression into (8.25) gives*

$$\begin{aligned} \frac{1}{\sqrt{n}} \sum_{l=1}^n x_l \mathbb{E}[\mathbf{W}_l \mathbf{Q}]_{ij} &= -\frac{1}{n}\mathbb{E}[\mathbf{Q}^2]_{ij} - \frac{1}{n}\mathbb{E}\left[\operatorname{tr} \mathbf{Q} \cdot [\mathbf{Q}]_{ij}\right] - \frac{1}{n}\mathbb{E}[\mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} \cdot [\mathbf{Q}]_{ij}] \\ &\quad - \frac{1}{n}\mathbb{E}\left[[\mathbf{Q}\mathbf{x}]_i [\mathbf{Q}\mathbf{x}]_j\right] - \frac{1}{n}\mathbb{E}\left[\operatorname{tr} \mathbf{Q} \cdot [\mathbf{Q}\mathbf{x}]_j\right] x_i - \frac{1}{n}\mathbb{E}[\mathbf{Q}^2\mathbf{x}]_j x_i \end{aligned}$$

where we used the fact that $\|\mathbf{x}\| = 1$*. Using* $\mathbf{Q} = -\frac{1}{z}\mathbf{I}_n + \frac{1}{z}\frac{1}{\sqrt{n}}\sum_{i=1}^n x_i \mathbf{W}_i \mathbf{Q}$*, we then find*

$$\begin{aligned} \mathbb{E}[\mathbf{Q}]_{ij} = -\frac{1}{z}\delta_{ij} - \frac{1}{z}\frac{1}{n}\mathbb{E}[\mathbf{Q}^2]_{ij} - \frac{1}{z}\frac{1}{n}\mathbb{E}\left[\operatorname{tr} \mathbf{Q} \cdot [\mathbf{Q}]_{ij}\right] - \frac{1}{z}\frac{1}{n}\mathbb{E}[\mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} \cdot [\mathbf{Q}]_{ij}] \\ - \frac{1}{z}\frac{1}{n}\mathbb{E}\left[[\mathbf{Q}\mathbf{x}]_i[\mathbf{Q}\mathbf{x}]_j\right] - \frac{1}{z}\frac{1}{n}\mathbb{E}\left[\operatorname{tr} \mathbf{Q} \cdot [\mathbf{Q}\mathbf{x}]_j\right] x_i - \frac{1}{z}\frac{1}{n}\mathbb{E}[\mathbf{Q}^2\mathbf{x}]_j x_i. \end{aligned}$$

This leads, with a concentration argument (using for instance the Nash-Poincaré inequality, Lemma 2.14, to show that the variances of $\frac{1}{n}\operatorname{tr} \mathbf{Q}$ *and* $\mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x}$ *vanish as* $n \to \infty$*), to*

$$\begin{aligned} \frac{1}{n}\operatorname{tr} \mathbf{Q} &= -\frac{1}{z} - \frac{1}{z}\left(\frac{1}{n}\operatorname{tr} \mathbf{Q}\right)^2 + O(n^{-1}) \\ \mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} &= -\frac{1}{z} - \frac{2}{z}\mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} \cdot \frac{1}{n}\operatorname{tr} \mathbf{Q} + O(n^{-1}). \end{aligned}$$

Since $\frac{1}{n}\operatorname{tr} \mathbf{Q}$ *satisfies in the* $n \to \infty$ *limit the fixed-point equation of the Wigner semicircle law (i.e.,* $m^2(z) + zm(z) + 1 = 0$*), the first part of the result unfolds. An interesting consequence of the results above is that, due to the existence of* $\mathbf{x}$ *in the expression of* $\mathbf{Y}_\mathbf{x}$*,* $\mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x}$ *does not satisfy the same equation as* $\frac{1}{n}\operatorname{tr} \mathbf{Q}$*. Precisely, we find that*

$$\begin{aligned} \frac{1}{n}\operatorname{tr} \mathbf{Q} &= -\frac{z}{2} + \frac{1}{2}\sqrt{z^2 - 4} + O(n^{-1}) \\ \mathbf{x}^\mathsf{T}\mathbf{Q}\mathbf{x} &= \frac{1}{\sqrt{z^2 - 4}} + O(n^{-1}) \end{aligned}$$

where the branch of the square root is chosen so that the right-hand side terms are Stieltjes transforms of (probability) measures. For further use, note that

$$\frac{\partial}{\partial z}\mathbf{x}^{\mathsf{T}}\mathbf{Q}(z)\mathbf{x} = -\frac{z}{(z^2-4)^{\frac{3}{2}}} + O(n^{-1}) \tag{8.26}$$

with the same convention on the square root.

As in Section 2.5 for spiked models, we write, for $(\hat{\lambda}_{\mathbf{x}}, \hat{\mathbf{u}}_{\mathbf{x}})$ an eigenvalue-eigenvector pair of $\mathbf{Y}_{\mathbf{x}}$,

$$\begin{aligned}
|\hat{\mathbf{u}}_{\mathbf{x}}^{\mathsf{T}}\mathbf{x}|^2 &= -\frac{1}{2\pi\imath}\oint_{\Gamma}\mathbf{x}^{\mathsf{T}}(\mathbf{Y}_{\mathbf{x}} - z\mathbf{I}_n)^{-1}\mathbf{x}\,dz \\
&= -\frac{1}{2\pi\imath}\oint_{\Gamma}\frac{\mathbf{x}^{\mathsf{T}}\mathbf{Q}\mathbf{x}}{1+\ell\mathbf{x}^{\mathsf{T}}\mathbf{Q}\mathbf{x}}\,dz
\end{aligned}$$

which holds for all large n with high probability over a positive contour Γ surrounding the limiting isolated eigenvalue $\lambda_{\mathbf{x}}$ (upon existence). By residue calculus, we find that this integral can only be non-vanishing if $\lambda_{\mathbf{x}}$ is a limiting root of $1+\ell\mathbf{x}^{\mathsf{T}}\mathbf{Q}(z)\mathbf{x}=0$, that is, if

$$\lambda_{\mathbf{x}} = \sqrt{\ell^2+4}$$

with (using for instance l'Hôpital's rule to evaluate the residue)

$$|\hat{\mathbf{u}}_{\mathbf{x}}^{\mathsf{T}}\mathbf{x}|^2 = -\frac{\mathbf{x}^{\mathsf{T}}\mathbf{Q}(\lambda_{\mathbf{x}})\mathbf{x}}{\ell\mathbf{x}^{\mathsf{T}}\mathbf{Q}'(\lambda_{\mathbf{x}})\mathbf{x}} + o(1) = \frac{\ell}{\sqrt{\ell^2+4}} + o(1)$$

as requested.

The proof is then concluded by remarking that, for $(\hat{\lambda}, \hat{\mathbf{u}})$ an "eigenvalue-eigenvector" pair of the tensor $\mathcal{Y}$ (as defined in (8.23)),

$$\begin{aligned}
\hat{\lambda}|\hat{\mathbf{u}}^{\mathsf{T}}\mathbf{x}| &= |\mathcal{Y}\cdot\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}\cdot\mathbf{x}| \leq \max_{\|\mathbf{v}\|=1}|\mathcal{Y}\cdot\mathbf{v}\cdot\mathbf{v}\cdot\mathbf{x}| \\
&= \max_{\|\mathbf{v}\|=1}\mathbf{v}^{\mathsf{T}}\mathbf{Y}_{\mathbf{x}}\mathbf{v} = \hat{\lambda}_{\mathbf{x}} = \lambda_{\mathbf{x}} + o(1)
\end{aligned}$$

which, in the limit, provides the upper bound

$$\hat{\lambda}|\hat{\mathbf{u}}^{\mathsf{T}}\mathbf{x}| \leq \sqrt{\ell^2+4}.$$