Probability and Stochastic Process II: Random Matrix Theory and Applications
Lecture 1: Introduction

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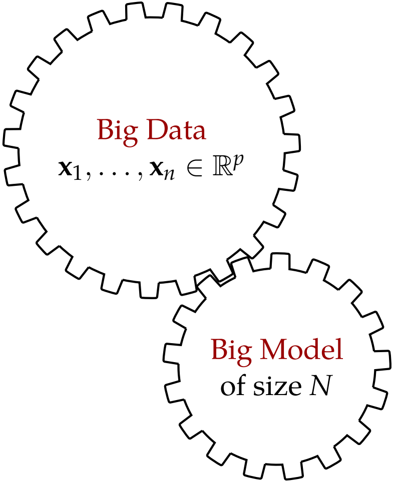
Outline



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- counterintuitive phenomena different from classical asymptotics statistics
- complete change of understanding of many methods in statistics, machine learning, signal processing, and wireless communications
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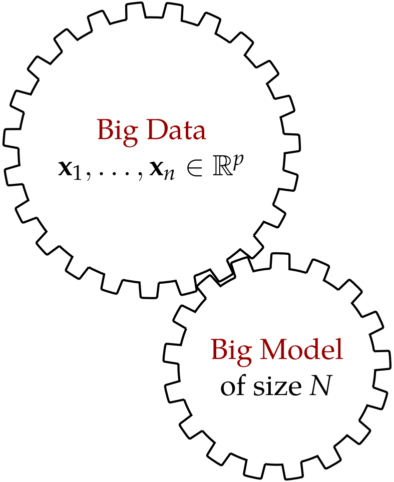

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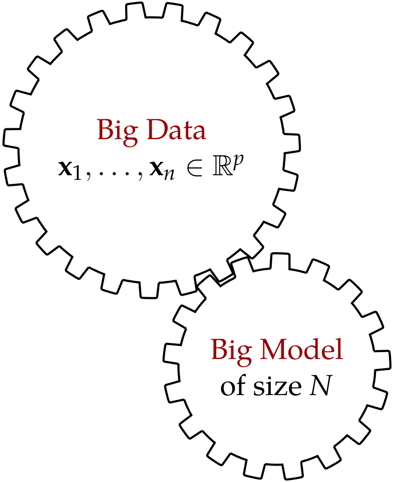

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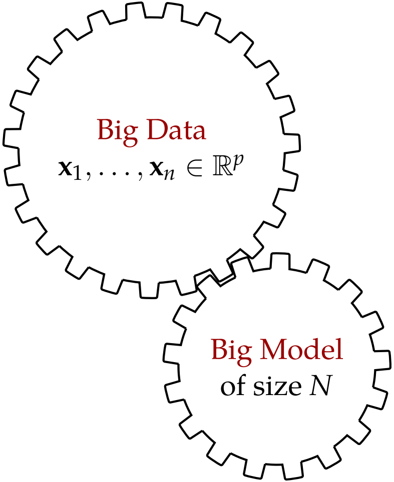

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Sample Covariance

RMT for Telecom

RMT for SP

RMT for ML

» **Problem**: estimate covariance $\mathbf{C} \in \mathbb{R}^{p \times p}$ from n data samples $\mathbf{x}_1, \ldots, \mathbf{x}_n$ with $\mathbf{x}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$,

» Maximum likelihood sample covariance matrix with entry-wise convergence

$$\hat{\mathbf{C}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^\mathsf{T} \in \mathbb{R}^{p \times p}, \quad [\hat{\mathbf{C}}]_{ij} \to [\mathbf{C}]_{ij}$$

almost surely as $n \to \infty$: optimal for $n \gg p$ (or, for p "small").

» In the regime $n \sim p$, conventional wisdom breaks down: for $\mathbf{C} = \mathbf{I}_p$ with $n < p$, $\hat{\mathbf{C}}$ has at least $p - n$ zero eigenvalues:

$$\boxed{\|\hat{\mathbf{C}} - \mathbf{C}\| \not\to 0, \quad n, p \to \infty} \Rightarrow \text{eigenvalue mismatch and not consistent!}$$

» due to $\|\mathbf{A}\|_\infty \leq \|\mathbf{A}\| \leq p\|\mathbf{A}\|_\infty$ for $\mathbf{A} \in \mathbb{R}^{p \times p}$ and $\|\mathbf{A}\|_\infty \equiv \max_{ij} |\mathbf{A}_{ij}|$.

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$$\mu(dx) = (1 - c^{-1})^+ \delta(x) + \frac{1}{2\pi cx} \sqrt{(x - E_-)^+ (E_+ - x)^+} dx$$

where $E_- = (1 - \sqrt{c})^2$, $E_+ = (1 + \sqrt{c})^2$ and $(x)^+ \equiv \max(x, 0)$. Close match!

Figure: Eigenvalue distribution of $\hat{\mathbf{C}}$ versus Marčenko-Pastur law, $p = 500$, $n = 50\,000$.

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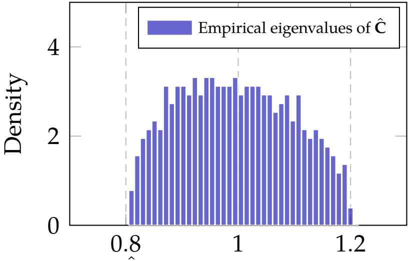

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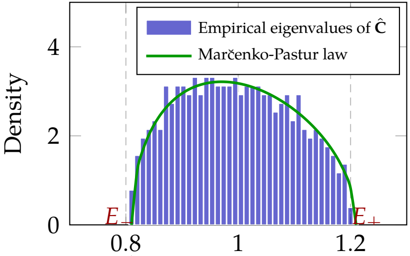

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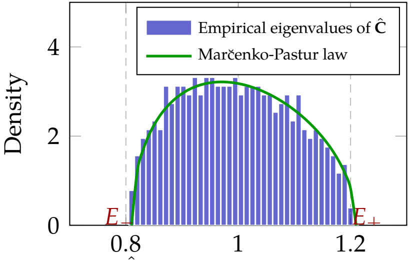

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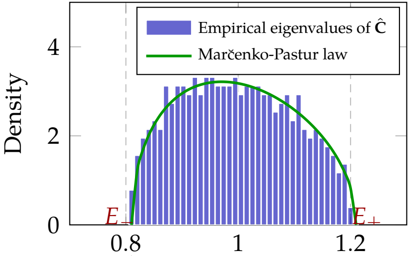

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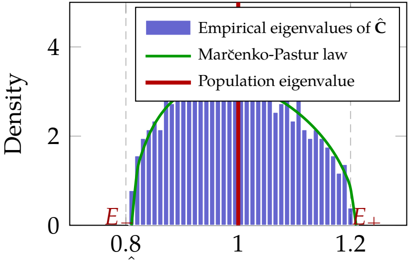

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Classical large-n asymptotic analysis mostly fails today

- large-n intuition, and many existing popular methods in biology, finance, signal processing, telecommunication, and machine learning, must **fail** even with $n = 100p$!
- **RMT** as a flexible and powerful tool to **understand** and **recreate** these methods
- in essence, "increasing complexity of the system models employed in above fields demand low complexity analysis"
- in the remainder, how RMT can be applied to assess
  - **telecommunication**: code division multiple access (CDMA) technology
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Application to telecom: performance analysis of CDMA via RMT

- CDMA in 3G succeeded the TDMA tech in 2G, for which users are successively allocated an exclusive amount of time to exchange data with the APs
- **major issue**: at the same time a very strict maximal number of users could be accepted by a given AP, *regardless* of the users' requests in terms of quality of service
- CDMA: to increase the max number of users, and to dynamically balancing the quality of service offered to each terminal
  - each user is allocated a (long) **spreading code** orthogonal to the other users' codes
  - so that all users can simultaneously receive data while experiencing a limited amount of interference from concurrent communications, due to code orthogonality
  - codes not fully orthogonal, more users served, more interference and then less quality of service; but at no time is a user rejected for lack of available resource
- **Question**: how to evaluate the **capacity** (max achievable transmission data rate) of CDMA network? (which clearly depends on pre-coding strategy)

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Orthogonal CDMA versus TDMA

For **orthogonal** CDMA, assume:

- frequency flat channel conditions for all users; and
- channel stability over a large number of successive symbol periods;

then the rates achieved in the up-link are maximal when the orthogonal codes are as long as the number of users n, with system capacity given by

$$C_{\rm orth}(\sigma^2) = \frac{1}{n} \log\det\left(\mathbf{I}_n + \frac{1}{\sigma^2}\mathbf{W}\mathbf{G}\mathbf{G}^{\sf H}\mathbf{W}^{\sf H}\right), \tag{1}$$

with noise power σ^2, $\mathbf{W} \in \mathbb{C}^{n\times n}$ the **orthogonal** CDMA codes ($\mathbf{W}$ unitary), and $\mathbf{G} \equiv \operatorname{diag}\{g_i\}_{i=1}^n$ represents channel **gains** of the users. Note that

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This justifies the equivalence between TDMA and **orthogonal** CDMA rate performance.

Orthogonal CDMA versus TDMA

For **orthogonal** CDMA, assume:

» frequency flat channel conditions for all users; and

» channel stability over a large number of successive symbol periods;

then the rates achieved in the up-link are maximal when the orthogonal codes are as long as the number of users n, with system capacity given by

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Random versus orthogonal CDMA

When it comes to (pseudo-)random CDMA with (random i.i.d. codes), under the same conditions, we have

$$C_{\rm rand}(\sigma^2) = \frac{1}{n} \log\det \left(\mathbf{I}_n + \frac{1}{\sigma^2} \mathbf{X}\mathbf{G}\mathbf{G}^{\sf H}\mathbf{X}^{\sf H} \right), \tag{3}$$

for $\mathbf{X} \in \mathbb{C}^{n\times n}$ the users' random codes.

Question: $C_{\rm rand}$ as a function of gains $\mathbf{G}$ and (distribution of) codes $\mathbf{X}$?

» (first?) answered by Shami, Tse, and Verdú in [5, 6];

» however capacity expressions not realistically achievable in practice, due to complicated and nonlinear processing algorithms;

» if only linear pre-coders and/or decoders are used, optimal solution:
  - Tse and Hanly in [4] for frequency flat channels;
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Outline

Signal sensing using multi-dimensional sensor arrays

Motivation:

- Shannon made us realize that, to achieve high rate of information transfer, increasing the transmission bandwidth is largely preferred over increasing the power
- high rate communications with finite power budget, need **frequency multiplexing**
- **cognitive radio**: to communicate not by exploiting the over-used frequency domain, or by exploiting the over-used space domain, but by exploiting so-called **spectrum holes**, jointly in time, space, and frequency

As such, a cognitive radio network (also called a *secondary network*)

- can help reuse the resources in a licensed (*first*) network
- but require constant **awareness** of the operations taking place in the licensed networks
- for example, via **signal sensing/detection**

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Hypothesis testing in a signal-plus-noise model for cognitive radios

System model: let $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \in \mathbb{R}^{p \times n}$ with i.i.d. columns $\mathbf{x}_i \in \mathbb{R}^p$ received by array of p sensors, signal decision as the following binary hypothesis test:

$$\mathbf{X} = \begin{cases} \sigma \mathbf{Z}, & \mathcal{H}_0 \\ \mathbf{a}\mathbf{s}^\mathsf{T} + \sigma \mathbf{Z}, & \mathcal{H}_1 \end{cases}$$

where $\mathbf{Z} = [\mathbf{z}_1, \ldots, \mathbf{z}_n] \in \mathbb{R}^{p \times n}$, $\mathbf{z}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_p)$, $\mathbf{a} \in \mathbb{R}^p$ deterministic of unit norm $\|\mathbf{a}\| = 1$, signal $\mathbf{s} = [s_1, \ldots, s_n]^\mathsf{T} \in \mathbb{R}^n$ with s_i i.i.d. random, and $\sigma > 0$. Denote $c = p/n > 0$.

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» If $\mathbf{a}$, σ, and statistics of s_i are known, the decision-optimal Neyman-Pearson () test:

$$\frac{\mathbb{P}(\mathbf{X} \mid \mathcal{H}_1)}{\mathbb{P}(\mathbf{X} \mid \mathcal{H}_0)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \alpha \tag{4}$$

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However,

- in practice, we do not know σ, nor the information vector $\mathbf{a} \in \mathbb{R}^p$ (to be recovered)
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Hypothesis testing in a signal-plus-noise model via GLRT

To set a maximum false alarm rate (or Type I error) of $r > 0$ for large n, p, according to RMT, one must choose a threshold $f(\alpha)$ for T_p:

$$\mathbb{P}(T_p \geq f(\alpha)) = r \Leftrightarrow \mu_{\mathrm{TW}_1}([A_p, +\infty)) = r, \quad A_p = (f(\alpha) - (1+\sqrt{c})^2)(1+\sqrt{c})^{-\frac{4}{3}} c^{\frac{1}{6}} n^{\frac{2}{3}} \tag{5}$$

with μ_{TW_1} the Tracy-Widom distribution in RMT.

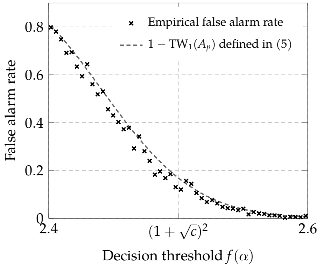

Figure: Comparison between empirical false alarm rates and $1 - \mathrm{TW}_1(A_p)$ for A_p of the form in (5), as a function of the threshold $f(\alpha) \in [(1+\sqrt{c})^2 - 5n^{-2/3}, (1+\sqrt{c})^2 + 5n^{-2/3}]$, for $p = 256$, $n = 1\,024$ and $\sigma = 1$.

Outline

"Curse of dimensionality": loss of relevance of Euclidean distance

» Binary Gaussian mixture classification $\mathbf{x} \in \mathbb{R}^p$:

$$\mathcal{C}_1 : \mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}_1, \mathbf{C}_1), \text{ versus } \mathcal{C}_2 : \mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}_2, \mathbf{C}_2);$$

» Neyman-Pearson test: classification is possible only when[a]

$$\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\| \geq C_\mu, \text{ or } \|\mathbf{C}_1 - \mathbf{C}_2\| \geq C_{\mathbf{C}} \cdot p^{-1/2}$$

for some constants $C_\mu, C_{\mathbf{C}} > 0$.

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as $n, p \to \infty$ (i.e., $n \sim p$), for $\mathbf{C}^\circ \equiv \frac{1}{2}(\mathbf{C}_1 + \mathbf{C}_2)$, regardless of the classes $\mathcal{C}_a, \mathcal{C}_b$!

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"Curse of dimensionality": loss of relevance of Euclidean distance

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Loss of relevance of Euclidean distance: visual representation

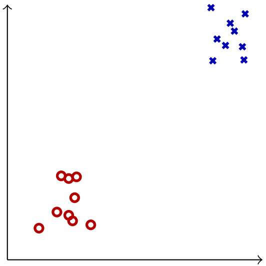
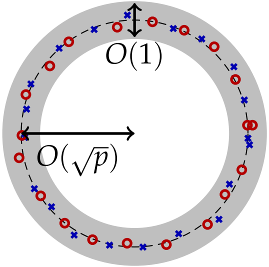

Figure: Visual representation of classification in (**left**) small and (**right**) large dimensions.

⇒ Direct consequence to various distance-based machine learning methods (e.g., kernel spectral clustering)!

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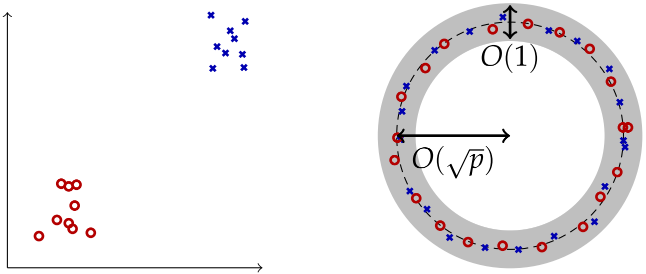

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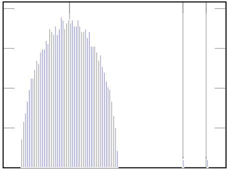
Reminder on kernel spectral clustering

Two-step classification of n data points with distance kernel $\mathbf{K} \equiv \{f(\|\mathbf{x}_i - \mathbf{x}_j\|^2/p)\}_{i,j=1}^n$:



⇓ **Top eigenvectors** ⇓

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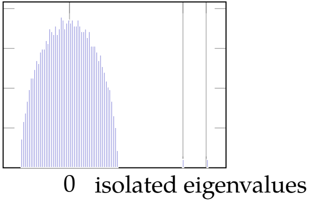

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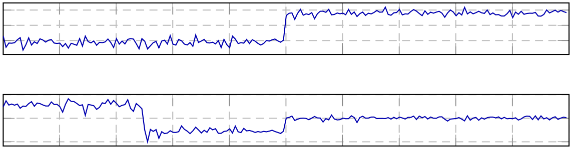

⇓ K-dimensional representation ⇓

⇓

EM or k-means clustering

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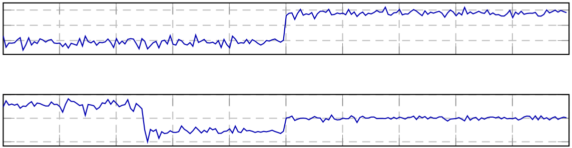

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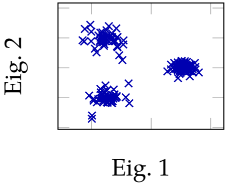

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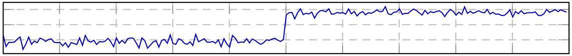

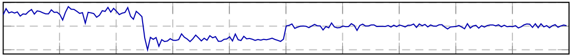

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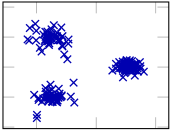
Eig. 2



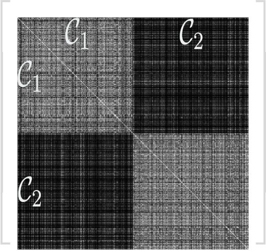
Eig. 1

⇓

EM or k-means clustering

Cluster Gaussian data $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbf{R}^p$ into $\mathcal{C}_1$ or $\mathcal{C}_2$, with second top eigenvectors $\mathbf{v}_2$ of heat kernel $\mathbf{K}_{ij} = \exp(-\|\mathbf{x}_i - \mathbf{x}_j\|^2/2p)$, small and large dimensional data.

(a) $p = 5, n = 500$

$\mathbf{K} =$ 

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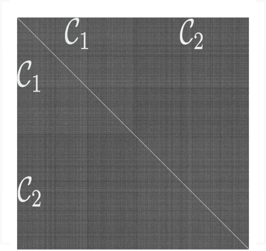
$\mathbf{K} =$ 

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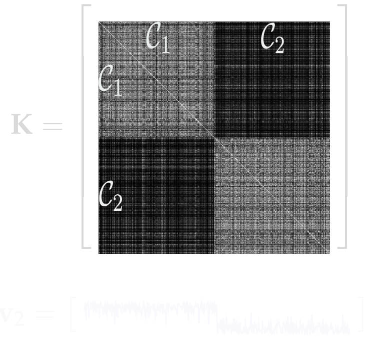

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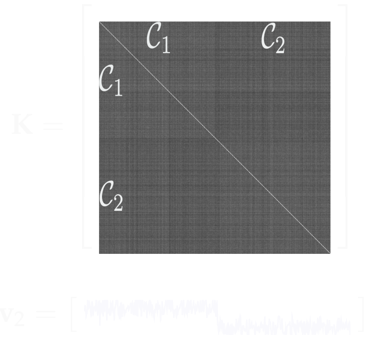

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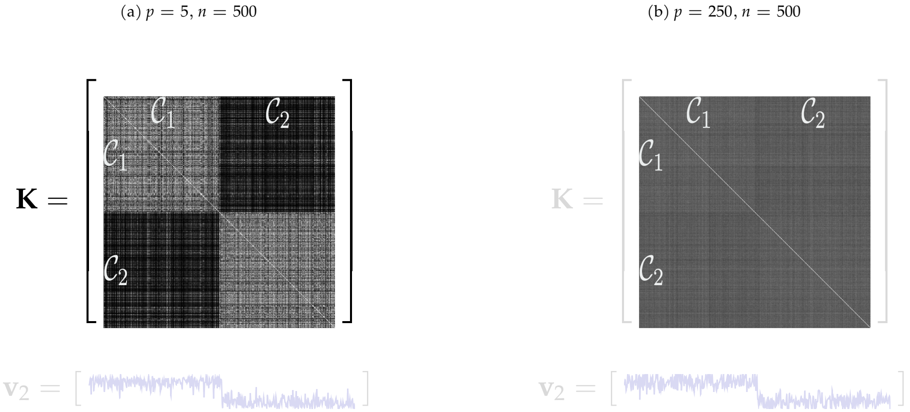

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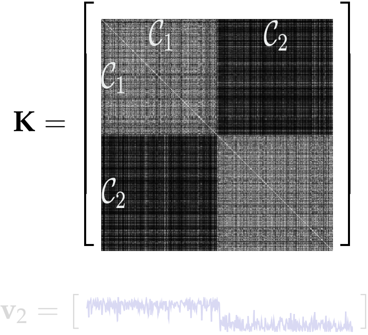

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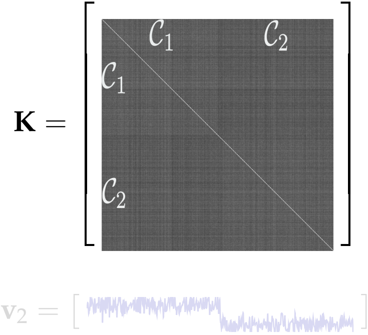

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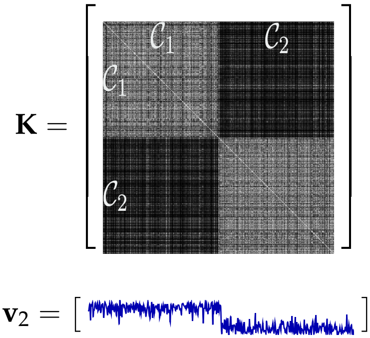

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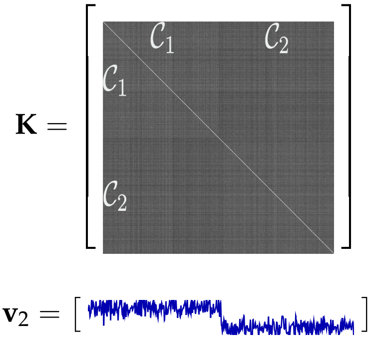
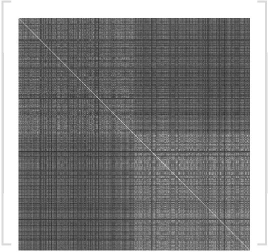


Kernel matrices for large dimensional real-world data

(a) MNIST

$\mathbf{K} =$ 

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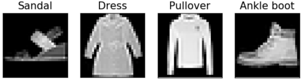
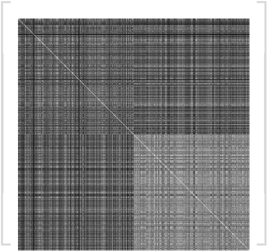
(b) Fashion-MNIST



$\mathbf{K} =$ 

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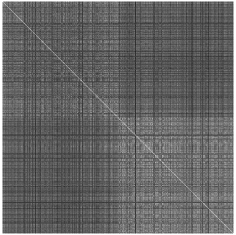
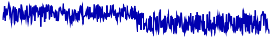
$\mathbf{K} =$ 

$\mathbf{v}_2 =$ 

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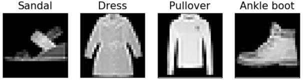
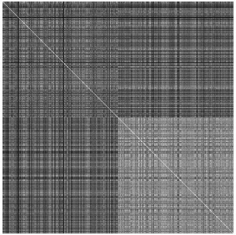
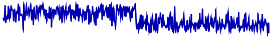


$\mathbf{K} =$ 

$\mathbf{v}_2 =$ 

A RMT viewpoint of large kernel matrices

» "local" linearization of *nonlinear* kernel matrices in large dimensions, e.g., Gaussian kernel matrix $\mathbf{K}_{ij} = \exp(-\|\mathbf{x}_i - \mathbf{x}_j\|^2/2p)$ with $\mathbf{C}_1 = \mathbf{C}_2 = \mathbf{I}_p$ (e.g., $\mathcal{C}_1 : \mathbf{x}_i = \boldsymbol{\mu}_1 + \mathbf{z}_i$ versus $\mathcal{C}_2 : \mathbf{x}_j = \boldsymbol{\mu}_2 + \mathbf{z}_j$) so that

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2/p \xrightarrow{a.s.} 2, \text{ and } \mathbf{K} = \exp\left(-\frac{2}{2}\right)\left(\mathbf{1}_n\mathbf{1}_n^\mathsf{T} + \frac{1}{p}\mathbf{Z}^\mathsf{T}\mathbf{Z}\right) + g(\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|)\frac{1}{p}\mathbf{j}\mathbf{j}^\mathsf{T} + * + o_{\|\cdot\|}(1)$$

with Gaussian $\mathbf{Z} = [\mathbf{z}_1, \ldots, \mathbf{z}_n] \in \mathbb{R}^{p\times n}$ and class-information $\mathbf{j} = [\mathbf{1}_{n/2}; -\mathbf{1}_{n/2}]$,

» accumulated effect of small "hidden" statistical information ($\|\mu_1 - \mu_2\|$ in this case)

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» **Same** phenomenon as the sample covariance example: $[\hat{\mathbf{C}} - \mathbf{C}]_{ij} \to 0 \not\Rightarrow \|\hat{\mathbf{C}} - \mathbf{C}\| \to 0$!

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Therefore

» **entry-wise**:

$$\mathbf{K}_{ij} = \exp(-1)\Big(1 + \underbrace{\frac{1}{p}\mathbf{z}_i^\mathsf{T}\mathbf{z}_j}_{O(p^{-1/2})}\Big) \pm \underbrace{\frac{1}{p}g(\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|)}_{O(p^{-1})} + *, \text{ so that } \frac{1}{p}g(\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|) \ll \frac{1}{p}\mathbf{z}_i^\mathsf{T}\mathbf{z}_j,$$

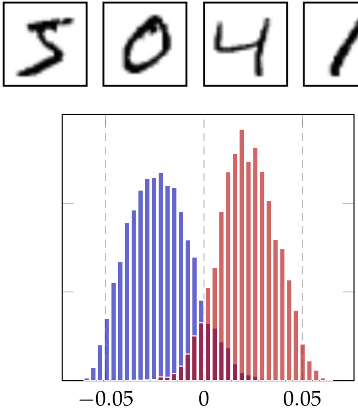
» **spectrum-wise**:

- $\|\mathbf{K} - \exp(-1)\mathbf{1}_n\mathbf{1}_n^\mathsf{T}\| \not\to 0$;
- $\|\frac{1}{p}\mathbf{Z}^\mathsf{T}\mathbf{Z}\| = O(1)$ and $\|g(\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|)\frac{1}{p}\mathbf{j}\mathbf{j}^\mathsf{T}\| = O(1)$!

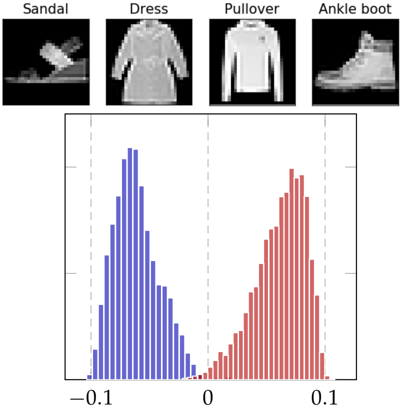
» **Same** phenomenon as the sample covariance example: $[\hat{\mathbf{C}} - \mathbf{C}]_{ij} \to 0 \not\Rightarrow \|\hat{\mathbf{C}} - \mathbf{C}\| \to 0$!

$\Rightarrow$ With **RMT**, we understand kernel spectral clustering for large dimensional data!

Some more numerical results

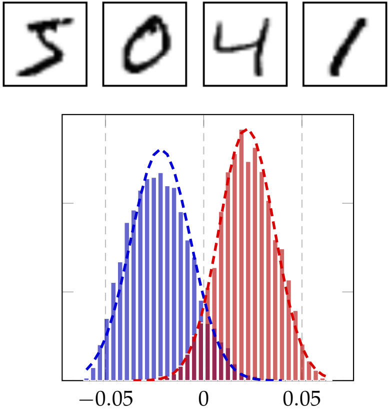


(a) MNIST

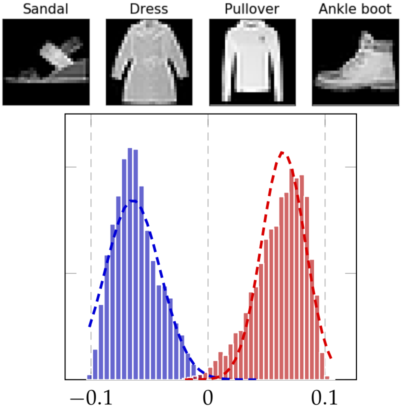


(b) Fashion-MNIST

Some more numerical results



(a) MNIST



(b) Fashion-MNIST

Thank you! And some more information

- Find more information in the monograph "Random Matrix Methods for Machine Learning" with Cambridge University Press
- with online book draft https://zhenyu-liao.github.io/pdf/RMT4ML.pdf
- with online code https://github.com/Zhenyu-LIAO/RMT4ML!
- and exercise solution https://zhenyu-liao.github.io/pdf/RMT4ML_solution.pdf

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Thank you! Q & A?

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