Probability and Stochastic Process II: Random Matrix Theory and Applications Lecture 5: Spiked Model

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April 5, 2023

Outline

What we will have today

- sample covariance matrix $\hat{\mathbf{C}} = \frac{1}{n}\mathbf{C}^{\frac{1}{2}}\mathbf{Z}\mathbf{Z}^\mathsf{T}\mathbf{C}^{\frac{1}{2}}$ and $\mathbf{C} = \mathbf{I}_p + \mathbf{P}$ for some low-rank matrix $\mathbf{P}$
- extreme eigenvalues of $\hat{\mathbf{C}}$ and connection to those of the low-rank $\mathbf{P}$
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- phase transition behavior and debiasing

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Low-rank update of SCM: eigenvalues

Low-rank update of SCM: eigenvectors

Limiting fluctuation

Applications

Low-rank perturbation of SCM

We have studied:

- spectral behavior of the SCM $\hat{\mathbf{C}} = \frac{1}{n}\mathbf{C}^{\frac{1}{2}}\mathbf{Z}\mathbf{Z}^{\mathsf{T}}\mathbf{C}^{\frac{1}{2}}$ for generic $\mathbf{C}$ and $\mathbf{Z}$ with i.i.d. entries
- in particular, how the eigenvalue distribution of $\hat{\mathbf{C}}$ (the previous μ) depends on that of $\mathbf{C}$ (denoted ν) and the dimension ratio $c = \lim p/n$
- characterization via implicit fixed point equation of the Stieltjes transform
- the behavior (e.g., location) of individual eigenvalue, however, remains unclear
- here, assess the behavior of individual eigenvalue and eigenvector via the **spiked model** analysis, in the *simple* setting of $\mathbf{C} = \mathbf{I}_p + \mathbf{P}$ with low rank $\mathbf{P}$

Note that the limiting eigenvalue distribution of $\hat{\mathbf{C}} = \frac{1}{n}(\mathbf{I}_p + \mathbf{P})^{\frac{1}{2}}\mathbf{Z}\mathbf{Z}^{\mathsf{T}}(\mathbf{I}_p + \mathbf{P})^{\frac{1}{2}}$ is in fact the same as that of $\frac{1}{n}\mathbf{Z}\mathbf{Z}^{\mathsf{T}}$, since the addition of low rank matrices asymptotically does not affect the **normalized** trace of the resolvent, and thus the Stieltjes transform.

Remark on the limiting eigenvalue distribution

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Low-rank perturbation from the Marčenko-Pastur law

Consider $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \in \mathbb{R}^{p \times n}$ with $\mathbf{x}_i = \mathbf{C}^{\frac{1}{2}} \mathbf{z}_i$, $\mathbf{z}_i \in \mathbb{R}^p$ with standard i.i.d. entries and

$$\mathbf{C} = \mathbf{I}_p + \mathbf{P}, \quad \mathbf{P} = \sum_{i=1}^{k} \ell_i \mathbf{u}_i \mathbf{u}_i^\mathsf{T}$$

with k and $\ell_1 \geq \ldots \geq \ell_k > 0$ fixed with respect to n, p.

- note that here $\nu \equiv \lim_{p \to \infty} \mu_{\mathbf{C}} = \lim_{p \to \infty} \frac{p-k}{p} \delta_1 + \frac{1}{p} \sum_{i=1}^{k} \delta_{1+\ell_i} = \delta_1$
- so, while $\mathbf{C} \neq \mathbf{I}_p$, the limiting μ still follows the Marčenko-Pastur law
- however, we do not have "no eigenvalue outside the support," since the condition $\operatorname{dist}(1 + \ell_i, \operatorname{supp}(\nu)) \not\to 0$ for $i \in \{1, \ldots, k\}$ is violated
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- note for $n \gg p$, $\hat{\mathbf{C}} \simeq \mathbf{C} = \mathbf{I}_p + \mathbf{P}$, so with its eigenvalues connected to those of $\mathbf{P}$

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Spiked eigenvalues and a phase transition

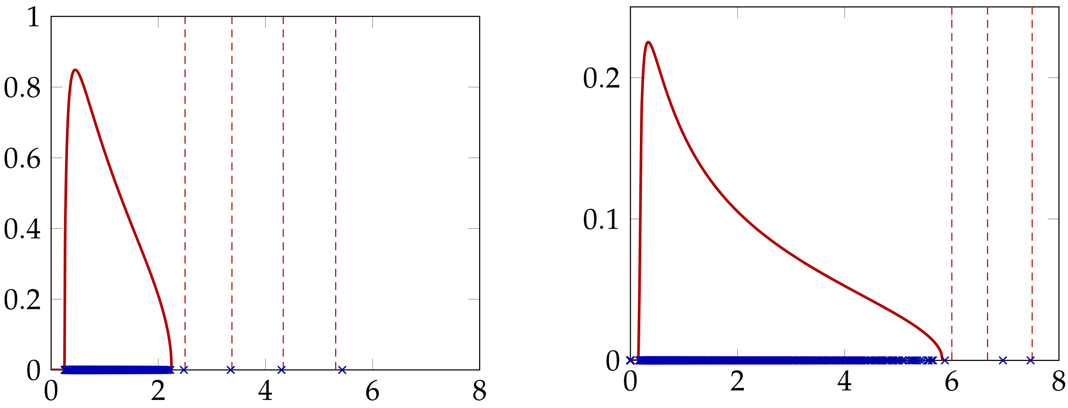
Here, depending on the values of ℓ_i and the ratio $c = \lim p/n$, the i-th largest eigenvalue $\hat{\lambda}_i$ of $\hat{\mathbf{C}}$ may indeed *isolate* from $\operatorname{supp}(\mu)$, due to [2].

For SCM $\hat{\mathbf{C}} = \frac{1}{n}\mathbf{C}^{\frac{1}{2}}\mathbf{Z}\mathbf{Z}^{\mathsf{T}}\mathbf{C}^{\frac{1}{2}}$ with i.i.d. $\mathbb{E}[\mathbf{Z}_{ij}^4] < \infty$, let $\mathbf{C} = \mathbf{I}_p + \mathbf{P}$ with $\mathbf{P} = \sum_{i=1}^k \ell_i \mathbf{u}_i \mathbf{u}_i^{\mathsf{T}}$ its spectral decomposition, where k and $\ell_1 \geq \ldots \geq \ell_k > 0$ are fixed with respect to n, p. Then, denoting $\hat{\lambda}_1 \geq \ldots \geq \hat{\lambda}_p$ the eigenvalues of $\hat{\mathbf{C}}$, as $n, p \to \infty$ with $p/n \to c \in (0, \infty)$,

$$\hat{\lambda}_i \xrightarrow{a.s.} \begin{cases} \lambda_i = 1 + \ell_i + c\frac{1+\ell_i}{\ell_i} > (1+\sqrt{c})^2 & , \ \ell_i > \sqrt{c} \\ (1+\sqrt{c})^2 & , \ \ell_i \leq \sqrt{c}. \end{cases}$$

Spiked eigenvalues

Jinho Baik and Jack W. Silverstein. “Eigenvalues of large sample covariance matrices of spiked population models”. In: *Journal of Multivariate Analysis* 97.6 (2006), 1382–1408. ISSN: 0047-259X. DOI: 10.1016/j.jmva.2005.08.003



(a) $p/n = 1/4$

(b) $p/n = 2$

Figure: Eigenvalues of $\frac{1}{n}\mathbf{X}\mathbf{X}^{\mathsf{T}}$ (**blue** crosses), the Marčenko-Pastur law (**red** solid line), and asymptotic spike locations (**red** dashed line), for $\mathbf{X} = \mathbf{C}^{\frac{1}{2}}\mathbf{Z}$, $\mathbf{C} = \mathbf{I}_p + \mathbf{P}$ with $\mu_{\mathbf{P}} = \frac{p-4}{p}\delta_0 + \frac{1}{p}(\delta_1 + \delta_2 + \delta_3 + \delta_4)$, for $p = 1024$ and different values of n.

Proof

- » solve the determinant equation $\det(\hat{\mathbf{C}} - \hat{\lambda}\mathbf{I}_p) = 0$ to find "isolated" eigenvalue $\hat{\lambda} \in \mathbb{R}$
- » use Sylvester's identity, $\det\left(\mathbf{AB} - \mathbf{I}_p\right) = \det\left(\mathbf{BA} - \mathbf{I}_k\right)$, to turn the p-dimensional equation into a k-dimensional one
- » solve the small-dimensional equation with the deterministic equivalent result

We write, with $\mathbf{X} = \mathbf{C}^{\frac{1}{2}}\mathbf{Z}$,

$$0 = \det\left(\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T} - \hat{\lambda}\mathbf{I}_p\right) = \det\left(\frac{1}{n}(\mathbf{I}_p + \mathbf{P})^{\frac{1}{2}}\mathbf{Z}\mathbf{Z}^\mathsf{T}(\mathbf{I}_p + \mathbf{P})^{\frac{1}{2}} - \hat{\lambda}\mathbf{I}_p\right)$$
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since $\det(\mathbf{I}_p + \mathbf{P}) \neq 0$. Note from the resolvent identity ($\mathbf{A}^{-1} - \mathbf{B}^{-1} = \mathbf{A}^{-1}(\mathbf{B} - \mathbf{A})\mathbf{B}^{-1}$)

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Proof (continue)

» We can then isolate the resolvent of the "whitened" model $\mathbf{Q}(\hat{\lambda}) = (\frac{1}{n}\mathbf{Z}\mathbf{Z}^\mathsf{T} - \hat{\lambda}\mathbf{I}_p)^{-1}$ and

$$0 = \det\left(\frac{1}{n}\mathbf{Z}\mathbf{Z}^\mathsf{T} - \hat{\lambda}\mathbf{I}_p + \hat{\lambda}(\mathbf{I}_p + \mathbf{P})^{-1}\mathbf{P}\right) = \det \mathbf{Q}^{-1}(\hat{\lambda}) \cdot \det\left(\mathbf{I}_p + \hat{\lambda}\mathbf{Q}(\hat{\lambda})(\mathbf{I}_p + \mathbf{P})^{-1}\mathbf{P}\right).$$

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For $\mathbf{Z} \in \mathbb{R}^{p\times n}$ with independent zero mean and unit variance random variables and $\mathbf{Q}(z) = (\frac{1}{n}\mathbf{Z}\mathbf{Z}^\mathsf{T} - z\mathbf{I}_p)^{-1}$, as $n, p \to \infty$ with $p/n \to (0,\infty)$, we have $\mathbf{Q}(z) \leftrightarrow \bar{\mathbf{Q}}(z) = m(z)\mathbf{I}_p$, with $m(z)$ the unique ST solution to $zcm^2(z) - (1-c-z)m(z) + 1 = 0$.

Deterministic equivalent result for SCM

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$$0 = \det\left(\mathbf{I}_k + \hat\lambda m(\hat\lambda)\cdot(\mathbf{I}_k+\mathbf{L})^{-1}\mathbf{L}\right) = \prod_{i=1}^k\left(1+\hat\lambda m(\hat\lambda)\frac{\ell_i}{1+\ell_i}\right). \quad (1)$$

» If such $\hat\lambda$ exists, must satisfy $\boxed{\hat\lambda m(\hat\lambda) = -\frac{1+\ell_i}{\ell_i} + o(1)}$, for some $i \in \{1,\ldots,k\}$.

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Proof (continue)

$$\boxed{\hat{\lambda} m(\hat{\lambda}) = -\frac{1+\ell_i}{\ell_i} + o(1)} \quad (2)$$

- For such a solution $\hat{\lambda}$ to exist, study the behavior of $xm(x) = \int \frac{x}{t-x}\mu(dt)$ which is increasing on its domain of definition with $xm(x) \to -1$ as $x \to \infty$.
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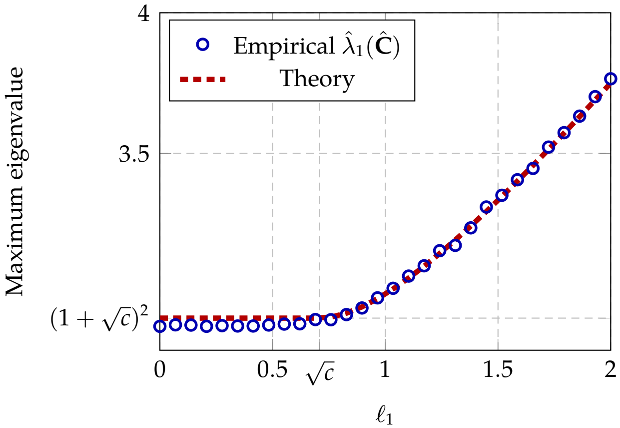

Figure: Phase transition behavior of the largest eigenvalue $\hat{\lambda}_1(\hat{\mathbf{C}})$ of $\hat{\mathbf{C}} = \frac{1}{n}(\mathbf{I}_p + \mathbf{P})^{\frac{1}{2}}\mathbf{Z}\mathbf{Z}^{\mathsf{T}}(\mathbf{I}_p + \mathbf{P})^{\frac{1}{2}}$ as a function of $\ell_1 = \|\mathbf{P}\|$ with rank one $\mathbf{P}$, for $p = 512$ and $n = 1\,024$. Empirical results obtained by averaging over 50 independent runs.

Outline

Spiked eigenvectors of SCM

We would also like to characterize the behavior of the **spiked eigenvectors**:

- it makes sense to believe that for $n \gg p$, $\hat{\mathbf{C}} \simeq \mathbf{C} = \mathbf{I}_p + \mathbf{P}$, so its eigenvectors should "close to" those of $\mathbf{C}$ in some way and for sufficient large n/p
- formally, how the **top** eigenvectors $\hat{\mathbf{u}}_1, \ldots, \hat{\mathbf{u}}_k$ of $\hat{\mathbf{C}}$ close to those $(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ of $\mathbf{P}$
- some type of phase transition behavior is (again) expected.

In the absence of low-rank perturbation $\mathbf{P}$ and Gaussian $\mathbf{Z}$, it is known that the eigenvectors of the resulting Wishart matrix $\frac{1}{n}\mathbf{Z}\mathbf{Z}^\mathsf{T} \in \mathbb{R}^{p \times p}$ are *uniformly distributed on the unit sphere* $\mathbb{S}^{p-1}$ (also know as the p-dimensional Haar measure), which is close to, **for** p **large**, random vector with i.i.d. Gaussian entries.

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Spiked eigenvector alignment

Let $\hat{\mathbf{u}}_1, \ldots, \hat{\mathbf{u}}_k$ be the eigenvectors associated with the largest k eigenvalues $\hat{\lambda}_1 > \ldots > \hat{\lambda}_k$ of $\hat{\mathbf{C}}$. Further assume that $\ell_1 > \ldots > \ell_k > 0$ are all distinct. Then, for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^p$ unit norm deterministic vectors

$$\mathbf{a}^\top \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i^\top \mathbf{b} - \mathbf{a}^\top \mathbf{u}_i \mathbf{u}_i^\top \mathbf{b} \cdot \frac{1 - c\ell_i^{-2}}{1 + c\ell_i^{-1}} \cdot 1_{\ell_i > \sqrt{c}} \xrightarrow{a.s.} 0. \tag{4}$$

In particular, with $\mathbf{a} = \mathbf{b} = \mathbf{u}_i$ we obtain

$$(\mathbf{u}_i^\top \hat{\mathbf{u}}_i)^2 \xrightarrow{a.s.} \zeta_i \equiv \frac{1 - c\ell_i^{-2}}{1 + c\ell_i^{-1}} \cdot 1_{\ell_i > \sqrt{c}}. \tag{5}$$

Spiked eigenvector alignment

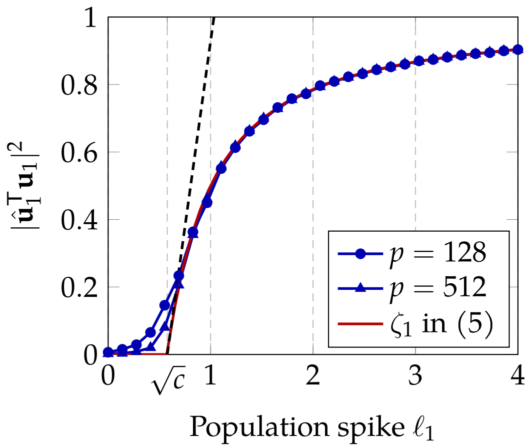

Figure: Empirical versus limiting $|\hat{\mathbf{u}}_1^\mathsf{T}\mathbf{u}_1|^2$ for $\mathbf{X} = \mathbf{C}^{\frac{1}{2}}\mathbf{Z}$, $\mathbf{C} = \mathbf{I}_p + \ell_1\mathbf{u}_1\mathbf{u}_1^\mathsf{T}$ and standard Gaussian $\mathbf{Z}$, $p/n = 1/3$, for different values of ℓ_1. Results obtained by averaging over 200 runs. In **black** dashed line the local behavior around $\sqrt{c}$.

Proof

» First write that, for all large n, p almost surely and $\ell_i > \sqrt{c}$,

$$\mathbf{a}^\mathsf{T}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i^\mathsf{T}\mathbf{b} = -\frac{1}{2\pi\imath}\oint_{\Gamma_{\lambda_i}} \mathbf{a}^\mathsf{T}\left(\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T} - z\mathbf{I}_p\right)^{-1}\mathbf{b}\, dz,$$

for Γ_{λ_i} a small contour enclosing only the almost sure limit $\lambda_i = 1 + \ell_i + c\frac{1+\ell_i}{\ell_i}$ of the eigenvalue $\hat{\lambda}_i$ of $\hat{\mathbf{C}}$ that we just determined.

$$\mathbf{a}^\mathsf{T}\left(\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T} - z\mathbf{I}_p\right)^{-1}\mathbf{b} = \mathbf{a}^\mathsf{T}\left(\frac{1}{n}(\mathbf{I}_p+\mathbf{P})^{\frac{1}{2}}\mathbf{Z}\mathbf{Z}^\mathsf{T}(\mathbf{I}_p+\mathbf{P})^{\frac{1}{2}} - z\mathbf{I}_p\right)^{-1}\mathbf{b}$$

$$= \mathbf{a}^\mathsf{T}(\mathbf{I}_p+\mathbf{P})^{-\frac{1}{2}}\left(\frac{1}{n}\mathbf{Z}\mathbf{Z}^\mathsf{T} - z\mathbf{I}_p + z(\mathbf{I}_p+\mathbf{P})^{-1}\mathbf{P}\right)^{-1}(\mathbf{I}_p+\mathbf{P})^{-\frac{1}{2}}\mathbf{b}$$

Proof (continue)

Denote $\mathbf{Q}(z) = (\frac{1}{n}\mathbf{Z}\mathbf{Z}^\mathsf{T} - z\mathbf{I}_p)^{-1}$, it follows from $(\mathbf{I}_p + \mathbf{P})^{-1} = \mathbf{I}_p - (\mathbf{I}_p + \mathbf{P})^{-1}\mathbf{P}$ and the spectral decomposition $(\mathbf{I}_p + \mathbf{P})^{-1}\mathbf{P} = \mathbf{U}(\mathbf{I}_k + \mathbf{L})^{-1}\mathbf{L}\mathbf{U}^\mathsf{T}$ for $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_k] \in \mathbb{R}^{p\times k}$ and $\mathbf{L} = \operatorname{diag}\{\ell_i\}_{i=1}^k$ that

$$
\begin{aligned}
&\mathbf{a}^\mathsf{T}\left(\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T} - z\mathbf{I}_p\right)^{-1}\mathbf{b} \\
&= \mathbf{a}^\mathsf{T}(\mathbf{I}_p + \mathbf{P})^{-\frac{1}{2}}\mathbf{Q}(z)(\mathbf{I}_p + \mathbf{P})^{-\frac{1}{2}}\mathbf{b} \\
&- z\mathbf{a}^\mathsf{T}(\mathbf{I}_p + \mathbf{P})^{-\frac{1}{2}}\mathbf{Q}(z)\mathbf{U}\left(\mathbf{I}_k + \mathbf{L}^{-1} + z\mathbf{U}^\mathsf{T}\mathbf{Q}(z)\mathbf{U}\right)^{-1}\mathbf{U}^\mathsf{T}\mathbf{Q}(z)(\mathbf{I}_p + \mathbf{P})^{-\frac{1}{2}}\mathbf{b} \\
&= \mathbf{a}^\mathsf{T}(\mathbf{I}_p + \mathbf{P})^{-\frac{1}{2}}\mathbf{Q}(z)(\mathbf{I}_p + \mathbf{P})^{-\frac{1}{2}}\mathbf{b} \\
&- z\mathbf{a}^\mathsf{T}(\mathbf{I}_p + \mathbf{P})^{-\frac{1}{2}}\mathbf{Q}(z)\mathbf{U}\left(\mathbf{L}^{-1} + (1 + zm(z))\mathbf{I}_k\right)^{-1}\mathbf{U}^\mathsf{T}\mathbf{Q}(z)(\mathbf{I}_p + \mathbf{P})^{-\frac{1}{2}}\mathbf{b} + o(1),
\end{aligned}
$$

where we used Woodbury identity and $\mathbf{U}^\mathsf{T}\mathbf{Q}(z)\mathbf{U} = m(z)\mathbf{I}_k + o_{\|\cdot\|}(1)$.

Proof (continue)

Objective: $\mathbf{a}^\mathsf{T}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i^\mathsf{T}\mathbf{b} = -\frac{1}{2\pi\imath}\oint_{\Gamma_{\lambda_i}} \mathbf{a}^\mathsf{T}\left(\frac{1}{n}\mathbf{X}\mathbf{X}^\mathsf{T} - z\mathbf{I}_p\right)^{-1}\mathbf{b}\,dz$, with $\mathbf{Q}(z) = (\frac{1}{n}\mathbf{Z}\mathbf{Z}^\mathsf{T} - z\mathbf{I}_p)^{-1}$ and

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Outline

Limiting fluctuations

- we have seen that the (asymptotic) **location** of the largest eigenvalue $\hat{\lambda}_1(\hat{\mathbf{C}})$ establishes a phase transition behavior if the corresponding population $\ell_i > \sqrt{c}$
- so below the threshold $\hat{\lambda}_1(\hat{\mathbf{C}}) = (1+\sqrt{c})^2 + o(1)$ almost surely as $n, p \to \infty$
- we want to understand more on this $o(1)$ local behavior

Under the same setting, assume $0 \le \ell_k < \ldots < \ell_1 < \sqrt{c}$. Then,

$$n^{\frac{2}{3}} \frac{\hat{\lambda}_1 - (1+\sqrt{c})^2}{(1+\sqrt{c})^{\frac{4}{3}} c^{-\frac{1}{6}}} \to \mathrm{TW}_1$$

in law, where TW_1 is the (real) Tracy-Widom distribution.

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- below phase transition: $\hat{\lambda}_1 = (1+\sqrt{c})^2 + n^{-\frac{2}{3}} T$ where T is a (scaled) Tracy-Widom RV

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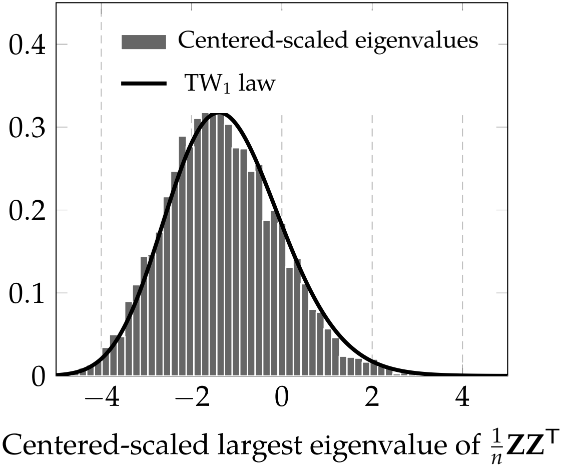

Figure: Empirical histogram of $n^{\frac{2}{3}} \frac{\hat{\lambda}_1 - (1+\sqrt{c})^2}{(1+\sqrt{c})^{\frac{4}{3}} c^{-\frac{1}{6}}}$ for $p = 256$, $n = 512$ and standard Gaussian $\mathbf{Z}$, versus the real Tracy-Widom law TW_1. Histogram obtained over 5 000 independent runs.

Remarks on Tracy-Widom law

- somewhat surprising: limiting fluctuation of $\hat{\lambda}_1$ is not Gaussian but follow the Tracy-Widom distribution and of order $O(n^{-2/3})$ (instead of $O(n^{-1/2})$ or $O(n^{-1})$)
- rate related to the following observation:
  - Marčenko-Pastur law: $\mu(dx) = \frac{1}{2\pi c x}\sqrt{(x-E_-)^+(E_+-x)^+}\,dx, \quad E_\pm = (1\pm\sqrt{c})^2.$
  - so near the right edge E_+: $\mu(dx) \simeq_{x\uparrow E_+} \frac{c^{1/4}}{\pi c(1+\sqrt{c})^2}\sqrt{|E_+-x|}$
  - so a typical number of eigenvalues in a space of size ϵ near the edge is

$$\int_{(1+\sqrt{c})^2-\epsilon}^{(1+\sqrt{c})^2} \sqrt{(1+\sqrt{c})^2 - x}\,dx \propto \epsilon^{\frac{3}{2}} \tag{6}$$

  - to have $O(1)$ eigenvalues within $[E_+-\epsilon, E_+]$ needs $\epsilon = O(n^{-\frac{2}{3}})$ (this is in fact the "spacing" between eigenvalues, which is of order $O(n^{-1})$ away from the edge)
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  - to have $O(1)$ eigenvalues within $[E_+-\epsilon, E_+]$ needs $\epsilon = O(n^{-\frac{2}{3}})$ (this is in fact the "spacing" between eigenvalues, which is of order $O(n^{-1})$ away from the edge)
- **Question**: hard-edge setting with $c = 1$, what happens?

Remarks on Tracy-Widom law

- somewhat surprising: limiting fluctuation of $\hat{\lambda}_1$ is not Gaussian but follow the Tracy-Widom distribution and of order $O(n^{-2/3})$ (instead of $O(n^{-1/2})$ or $O(n^{-1})$)
- rate related to the following observation:
  - Marčenko-Pastur law: $\mu(dx) = \frac{1}{2\pi c x}\sqrt{(x-E_-)^+(E_+-x)^+}\,dx, \quad E_\pm = (1 \pm \sqrt{c})^2$.
  - so near the right edge E_+: $\mu(dx) \simeq_{x \uparrow E_+} \frac{c^{1/4}}{\pi c (1+\sqrt{c})^2}\sqrt{|E_+ - x|}$
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Remarks on Fluctuation of the largest eigenvalue

- **BBP phase transition** [1]: named after the authors Jinho Baik, Gerard Ben Arous, Sandrine Peche, says that beyond the phase transition threshold, the fluctuation becomes a standard CLT type of order $O(n^{-1/2})$, from TW law of order $O(n^{-2/3})$
- universality for Tracy-Widom, real TW_1, complex TW_2, and quaternionic TW_4, for Wishart and Wigner matrix models, smallest and largest eigenvalues
- Tracy–Widom distributions connected in the asymptotics of a few growth models in the Kardar–Parisi–Zhang (KPZ) universality class

More spiked models:

- information-plus-noise model of the type $\frac{1}{n}(\mathbf{Z}+\mathbf{P})(\mathbf{Z}+\mathbf{P})^{\mathsf{T}}$
- additive $\mathbf{X}+\mathbf{P}$ for Wishart or Wigner type $\mathbf{X}$

Jinho Baik, Gérard Ben Arous and Sandrine Péché. "Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices". In: *The Annals of Probability* 33.5 (2005), 1643–1697. ISSN: 0091-1798. DOI: 10.1214/009117905000000233

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Outline

Hypothesis testing in a signal-plus-noise model for cognitive radios

System model: let $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \in \mathbb{R}^{p \times n}$ with i.i.d. columns $\mathbf{x}_i \in \mathbb{R}^p$ received by array of p sensors, signal decision as the following binary hypothesis test:

$$\mathbf{X} = \begin{cases} \sigma \mathbf{Z}, & \mathcal{H}_0 \\ \mathbf{a}\mathbf{s}^\mathsf{T} + \sigma \mathbf{Z}, & \mathcal{H}_1 \end{cases}$$

where $\mathbf{Z} = [\mathbf{z}_1, \ldots, \mathbf{z}_n] \in \mathbb{R}^{p \times n}$, $\mathbf{z}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_p)$, $\mathbf{a} \in \mathbb{R}^p$ deterministic of unit norm $\|\mathbf{a}\| = 1$, signal $\mathbf{s} = [s_1, \ldots, s_n]^\mathsf{T} \in \mathbb{R}^n$ with s_i i.i.d. random, and $\sigma > 0$. Denote $c = p/n > 0$.

» observation of either zero-mean Gaussian **noise** $\sigma \mathbf{z}_i$ of power σ^2, or deterministic **information** vector $\mathbf{a}$ modulated by an added scalar (random) **signal** s_i (e.g., ± 1).

» If $\mathbf{a}$, σ, and statistics of s_i are known, the decision-optimal Neyman-Pearson () test:

$$\frac{\mathbb{P}(\mathbf{X} \mid \mathcal{H}_1)}{\mathbb{P}(\mathbf{X} \mid \mathcal{H}_0)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \alpha \tag{7}$$

for some $\alpha > 0$ controlling the Type I and II error rates.

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Hypothesis testing via GLRT

- However, in practice, we do not know σ, nor the information vector $\mathbf{a} \in \mathbb{R}^p$ (to be recovered)
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Hypothesis testing in a signal-plus-noise model via GLRT

To set a maximum false alarm rate (or Type I error) of $r > 0$ for large n, p, according to RMT, one must choose a threshold $f(\alpha)$ for T_p:

$$\mathbb{P}(T_p \geq f(\alpha)) = r \Leftrightarrow \mu_{\mathrm{TW}_1}([A_p, +\infty)) = r, \quad A_p = (f(\alpha) - (1+\sqrt{c})^2)(1+\sqrt{c})^{-\frac{4}{3}} c^{\frac{1}{6}} n^{\frac{2}{3}} \quad (8)$$

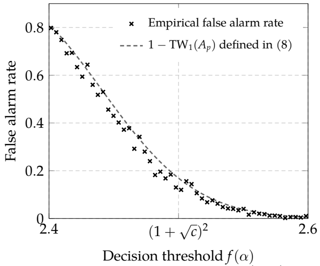
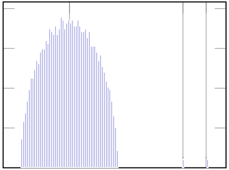


Figure: Comparison between empirical false alarm rates and $1 - \mathrm{TW}_1(A_p)$ for A_p of the form in (8), as a function of the threshold $f(\alpha) \in [(1+\sqrt{c})^2 - 5n^{-2/3}, (1+\sqrt{c})^2 + 5n^{-2/3}]$, for $p = 256$, $n = 1\,024$ and $\sigma = 1$.

Reminder on kernel spectral clustering

Two-step classification of n data points with distance kernel $\mathbf{K} \equiv \{f(\|\mathbf{x}_i - \mathbf{x}_j\|^2/p)\}_{i,j=1}^n$:



⇓ **Top eigenvectors** ⇓

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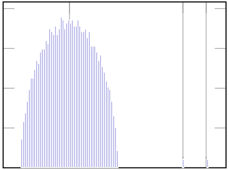

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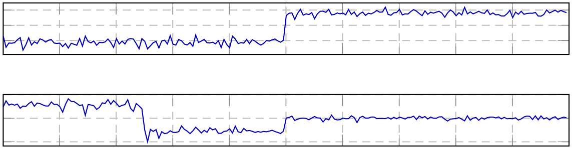

⇓ K-dimensional representation ⇓

⇓

EM or k-means clustering

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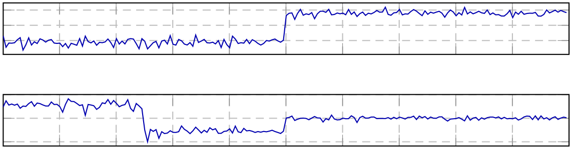

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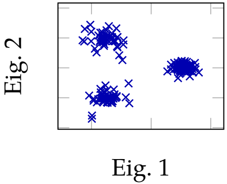

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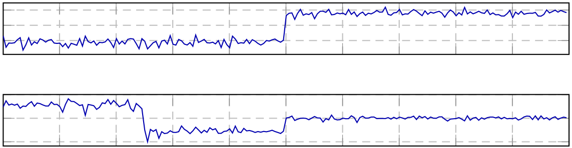

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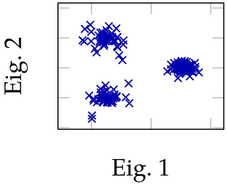

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EM or k-means clustering

Application to "compressed" spectral clustering

Entry-wise *nonlinear* transformation of $\mathbf{X}^\mathsf{T}\mathbf{X}$: $\mathbf{K} = \{f(\mathbf{x}_i^\mathsf{T}\mathbf{x}_j/\sqrt{p})/\sqrt{p}\}_{i,j=1}^n$, with

Sparsification: $f_1(t) = t \cdot 1_{|t|>\sqrt{2}s}$

Quantization: $f_2(t) = 2^{2-M}(\lfloor t \cdot 2^{M-2}/\sqrt{2}s\rfloor + 1/2) \cdot 1_{|t|\le\sqrt{2}s} + \mathrm{sign}(t) \cdot 1_{|t|>\sqrt{2}s}$

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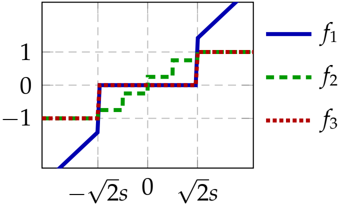

Notations: For each f and $\xi \sim \mathcal{N}(0,1)$, define the (generalized) moments

$$a_0 = \mathbb{E}[f(\xi)] = 0, \quad \mathbf{a_1} = \mathbb{E}[\xi f(\xi)], \quad \mathbf{a_2} = \mathbb{E}[\xi^2 f(\xi)]/\sqrt{2}, \quad \boldsymbol{\nu} = \mathbb{E}[f^2(\xi)] \geq a_1^2 + a_2^2. \tag{9}$$

f	$\mathbf{a_1}$	$\boldsymbol{\nu}$
f_1	$\operatorname{erfc}(s) + 2se^{-s^2}/\sqrt{\pi}$	$\operatorname{erfc}(s) + 2se^{-s^2}/\sqrt{\pi}$
f_2	$\sqrt{\frac{2}{\pi}} \cdot 2^{1-M}(1 + e^{-s^2} + \sum_{k=1}^{2^{M-2}-1} 2e^{-\frac{k^2 s^2}{4^{M-2}}})$	$1 - \frac{2^M-1}{4^{M-1}} \operatorname{erf}(s) - \sum_{k=1}^{2^{M-2}-1} \frac{k \operatorname{erf}(ks \cdot 2^{2-M})}{2^{2M-5}}$
f_3	$e^{-s^2}\sqrt{2/\pi}$	$\operatorname{erfc}(s)$

with $\mathbf{a_2} = \mathbf{0}$, $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$ error/complementary error function.

As $n, p \to \infty$ with $p/n \to c \in (0, \infty)$, the empirical spectral measure $\omega_{\mathbf{K}} = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i(\mathbf{K})}$ can be asymptotically determined by $m(z) = \int (t-z)^{-1} \omega(dt)$ solution to

$$z = -\frac{1}{m(z)} - \frac{\nu - {a_1}^2}{c} m(z) - \frac{{a_1}^2 m(z)}{c + a_1 m(z)}. \tag{10}$$

Limiting spectral measure

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Limiting spectral measure

For $a_1 > 0$ and $\mathbf{a_2 = 0}$, similarly define $F(x) = x^4 + 2x^3 + \left(1 - \frac{c\nu}{{a_1}^2}\right)x^2 - 2cx - c$ and $G(x) = \frac{a_1}{c}(1+x) + \frac{a_1}{x} + \frac{\nu - {a_1}^2}{a_1}\frac{1}{1+x}$ and let γ be the largest real solution to $F(\gamma) = 0$. Then,

$$\hat{\lambda} \to \lambda = \begin{cases} G(\rho), & \rho > \gamma \\ G(\gamma), & \rho \leq \gamma \end{cases}, \quad \frac{1}{n}|\hat{\mathbf{v}}^\mathsf{T}\mathbf{v}|^2 \to \alpha = \begin{cases} \frac{F(\rho)}{\rho(1+\rho)^3}, & \rho > \gamma \\ 0, & \rho \leq \gamma \end{cases} \tag{11}$$

as $n, p \to \infty$ with $p/n \to c \in (0, \infty)$, for SNR $\rho = \lim \|\boldsymbol{\mu}\|^2$.

Informative spike and a phase transition

"Compressed" spectral clustering: practical implications

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$$\frac{1}{n}\sum_{i=1}^{n} \delta_{\hat{\mathcal{C}}_i \neq \mathcal{C}_i} \to \frac{1}{2}\operatorname{erfc}(\sqrt{\alpha/(2-2\alpha)}), \quad \alpha = \lim_{n,p\to\infty} \frac{1}{n}|\hat{\mathbf{v}}^\mathsf{T}\mathbf{v}|^2.$$

Performance of spectral clustering

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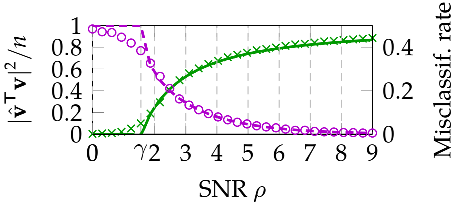

Consequence: optimal quantization/binarization threshold